Complex Analysis Exercise 10

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1. Let $f : D := \{|z| = 1\} \to \mathbb{C}$ be a continuous function. Assume that there exists a family of polynomials $P_n \to f$ which converges to f uniformly on D. Prove that there exists $F : D \to \mathbb{C}$ which is continuous and holomorphic at interior points such that $F|_D = f$,

2. Compute Taylor expansion around zero for the following functions:

(a) $z\,cos^2 z$

(b) $sinh(z)$

(c) $\frac{5z-1}{z^2-2z-15}$

(d) $\frac{1}{(1-z)^3}$

(e) $Log\frac{1+iz}{1-iz}$.

3. (a)Calculate Taylor expansion of $\frac{1}{1-z}$ around points $0, i, -1$. Find radius of convergence in each case.
(b) Let $z_0 \in \mathbb{C}^*$. Find the series expansion of $f(z) = \frac{1}{z}$ around z_0 and determine its radius of convergence. (c) Show that the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$ is equal to e and that the series diverges everywhere on the boundary.

4. Consider the series

$$arcsin(z) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{z^{2n+1}}{2n+1}.$$

Determine its radius of convergence and show that $arcsin(z)$ is the unique inverse function of $sin(z)$, wherever it is defined.

5. Let $D = \{z||z| < 1\}$ be a unit disk and $\overline{D}$ be the closure. Give an example of a continuous function $f : \overline{D} \to \mathbb{C}$ that is holomorphic on D, but does not have a holomorphic continuation on any domain in $\mathbb{C}$ containing $\overline{D}$.