Complex Analysis Exercise 12

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1. Let A be a square centered at the origin. Denote by s one of sides of A. Let $f : A \to \mathbb{C}$ be holomorphic on interior of A and continuous on the boundary of A such that $f(z) = 0$ for all $z \in s$. Prove that $f = 0$ on A.

2. Let f be an entire function which satisfies the following property: $\forall z \in \mathbb{C}$, there exists $N = N(z)$ (not necessarily constant) such that $f^{(N)}(z) = 0$. Prove that f is a polynomial.

3. Let f be an entire function. Assume that $f(z) \in \mathbb{R}$ for any $z \in (-\frac{1}{2019}, \frac{1}{2019})$. Show that $f(\bar{z}) = \overline{f(z)}$.

4. Let f be holomorphic in the unit disk. Assume that f satisfies $f(\frac{1}{2k}) = f(\frac{1}{2k+1})$, $\forall k \in \mathbb{N}$. Show that f is constant.

5. Find all singular points for the following functions. Determine their type and compute the residues:

(a) $\frac{1}{z^3 - z^5}$

(b) $\frac{e^z - 1}{z^n}$, $n \geq 1$

(c) $sin\, z \cdot sin\, \frac{1}{z}$

(d) $\frac{1}{e^z + 1}$

(e) $\frac{1}{sin^2(z)}$

(f) $\frac{z}{1 - e^{z^2}}$.

6. Let f, g be entire and non-constant functions. Assume that $|f(z)| \leq |g(z)|$ for all z. Prove that there exists $c \in \mathbb{C}$ such that $f(z) = cg(z)$.