Complex Analysis Exercise 13

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1. Let f be a holomorphic function with essential singularity at z_0. Assume that z_0 remains isolated singularity for $\frac{1}{f(z)}$. Prove that z_0 is essential singularity for $\frac{1}{f(z)}$.

2. Let f be holomorphic in the punctured disk $D^*_{z_0}(r) = \{r > |z - z_0| > 0\}$. Assume that f has no real values. Prove that z_0 is a removable singularity.

3. Let f be a function such that
   (a) f has finite number of singular points and they are all poles.
   (b) $f(z^{-1})$ has finite number of singular points and they are all poles.
Show that f is a rational function. Namely, $f(z) = \frac{p(z)}{q(z)}$, $p, q \in \mathbb{C}[z]$.

4. Let $f : D^*_{z_0}(r) \to \mathbb{C}$ be a holomorphic function. Assume that z_0 is not a removable singularity. Show that $e^{f(z)}$ has an essential singularity at z_0.

5. Let $f(z) = \sum_{k=0}^{\infty} \frac{z^k}{k^2}$. One can check the following properties of $f(z)$:

   (a) The radius of convergence is 1 and it has a singular point on $\{|z| = 1\}$.

   (b) It is not a pole.

   (c) There is no essential singularity.

Does this contradict to the theory?

6. Compute the following integral:

$$\int_0^{\infty} \frac{sin\alpha x}{x(x^2 + \beta)} dx, \ \alpha, \beta > 0.$$

7. Let $z_1, \cdots, z_r \in \mathbb{C}$ be distinct points. Let $f : \mathbb{C} \setminus \{z_1, \cdots, z_r\} \to \mathbb{C}$ be a holomorphic function such that $\lim_{|z| \to \infty} f(z) = 0$. Prove that

$$\sum_{i=1}^{r} Res_{z_i}(f) = \lim_{|z| \to \infty} z.f(z).$$