Complex Analysis Exercise 4

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1. Prove or give a counterexample of the following statements:
   (a) $Log(zw) = Log(z) + Log(w)$
   (b) $Log(z^{-1}) = -Log(z)$, $z \neq 0$
   (c) $z^{w+1} = z \cdot z^w$.

2. Solve the equation $cos(z) = 0$, $z \in \mathbb{C}$.

3. Calculate the integral $\int_\gamma Rez + Imzdz$ where $\gamma(t)$ is given by:
   (a) $t + it^2$, $t \in [-1, 1]$
   (b) $1 + t + i(2 + t)$, $t \in [0, 1]$
   (c) e^{it}, $t \in [0, 2\pi]$.

4. Compute the following path integrals:
   (a) $\int_\gamma cos(Rez)dz$ where γ is a circle around i with radius 1 with counter-clockwise orientation.
   (b) $\int_\gamma \frac{Logz}{z} dz$ $\gamma(t) = e^{it}$, $t \in [0, \pi]$.
   (c) $\int_\gamma (\overline{z})^n dz$ for any $n \in \mathbb{Z}$ and γ where γ is the unit circle with counter-clockwise orientation.

5. For any integer $n \geq 1$, prove

$$\int_0^{2\pi} cos^{2n} t dt = 2^{1-2n} \binom{2n}{n} \pi.$$

6. Give an example of a path $\gamma : [0, 1] \to \mathbb{C}$ of infinite length.

7. Let $MNPQ$ be a rectangle on the complex plane whose sides are parallel to the x, y-axes. It is divided into smaller rectangles whose sides are parallel to the axes as well. It is known that each smaller rectangle has at least one side (horizontal or vertical) whose length belongs to integers. Prove that $MNPQ$ also has at least one side of integer length.

8. Let $D \subset \mathbb{C}$ be a domain. Let $\gamma : [a, b] \to D$ be a piecewise regular path such that $f \circ \gamma$ is continuous. Prove that

$$\int_{\gamma} f(z)dz = \int_{a}^{b} f(\gamma(t))\dot{\gamma}(t)dt.$$