Complex Analysis Exercise 5

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1. Compute the following integrals:

   (a) $\int_{|z|=3} \frac{z}{(z-1)(z-i)} dz$,

   (b) $\int_{|z|=2} \frac{e^z}{z^2-1} dz$,

   (c) $\int_\gamma 2z - 3\overline{z} + 1 dz$ where γ is the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$,

   (d) $\int_\gamma \frac{dz}{\sqrt{z}}$ ($\sqrt{z} = e^{\frac{1}{2}Logz}$), where $\gamma = \{e^{it}|0 \leq t \leq \pi\}$.

2. (a) Show that for any rational function $R(x)$ which is defined on $[-1, 1]$, the following holds:

$$\int_0^{2\pi} R(cos\theta)d\theta = \int_{|z|=1} R(\frac{1}{2}(z + z^{-1}))\frac{dz}{iz}$$

(b) Compute $\int_0^{2\pi} \frac{d\theta}{a+cos\theta}$ for $a > 1$.

3. Compute $\int_0^t xsin(2x)dx$ using the complex integral $\int_{[0,t]} ze^{2iz}dz$.

4. (a) Liouville's theorem says that every bounded entire function is constant. Prove the theorem in the following way: Let f be an entire and bounded function. Pick $a \neq b \in \mathbb{C}$. Let R be a real number greater than $|a|$ and $|b|$. Calculate $\int_{|z|=R} \frac{f(z)dz}{(z-a)(z-b)}$ and check what happens when $R \to \infty$.
(b) Let f be an entire function with two periods. Show that f is constant.
(c) Prove the fundamental theorem of algebra: Let p be a nonconstant polynomial with complex coefficients. Then there exists $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.
(d) Find all entire function f such that $|f'(z)| < e^{-(Rez)^2}$ for all $z \in \mathbb{C}$.

5. Let $D \subset \mathbb{C}$ be a unit disk at the origin. Find all functions $f(z)$ which are holomorphic on D and which satisfy

$$f(\frac{1}{n}) = n^2 f(\frac{1}{n})^3, \quad n = 2, 3, 4, \cdots$$