Complex Analysis Exercise 7

Prof. Dr. Paul Biran

Due: 01.11.2019

1. Let k be a field and $p(x)$ be a polynomial with coefficients in k. If $p(x_0) = 0$ for some $x_0 \in k$, show that $p(x) = (x - x_0)q(x)$ for some polynomial $q(x)$.

2. Use Cauchy estimates to prove the following statement: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that there exsits $n \in \mathbb{N}$ and $R, C > 0$ such that

$$|f(z)| < C|z|^n, \text{ for any } |z| > R.$$

Then f is a polynomial in z whose degree is less than or equal to n.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

4. Prove that there is not entire function f such that $\forall z \in \mathbb{C}$, $|f(z)| > |z|$.

5. Let f be an entire function. Prove that in each of the following cases, f is constant:

(a) f satisfies $Im(f(z)) \leq 0$ for all $z \in \mathbb{C}$

(b) $|f(z)| \neq 1$ for all $z \in \mathbb{C}$

(c) f does not receive any value in $\mathbb{R}^- = \{x \in \mathbb{R} | x \leq 0\}$.

6. Let $\mathcal{D} = \{z \in \mathbb{C} | |z| < 1\}$ be a unit disk. Let f be a holomorphic function on the unit disk. Assume that $|f(z)| \leq |f(z^2)|$ in $\mathcal{D}$. Prove that f is constant.

7. Let $\mathcal{D}$ be the open disk as above. Find all biholomorphic function $f : \mathcal{D} \to \mathcal{D}$.