Complex Analysis Exercise 9

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1. Let $\gamma : [a, b] \to \mathbb{C}$ and $\lambda : [b, c] \to \mathbb{C}$ be two C^1 curves such that $\gamma(b) = \lambda(b)$. Prove that there exists a reparametrization of the curve $\gamma + \lambda$ which is C^1 (Here we allow a reparametrization such that $\gamma'(t_0) = 0$)

2. (a) Prove that the sequence $f_n(z) = z^n$, $n \geq 1$ converges locally uniformly but not uniformly on $\{z : |z| < 1\}$.
(b) Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary (not necessarily continuous) function and for $n \in \mathbb{N}$ define $f_n : \mathbb{C} \to \mathbb{C}$

$$f_n(z) = \begin{cases} f(z) & \text{if } |z| \leq n, \\ 0, & \text{if } |z| > n. \end{cases}$$

Show that the sequence (f_n) converges pointwise and locally uniformly to f, and that it converges uniformly to f, if and only if $\lim_{|z|\to\infty} f(z) = 0$.
(c) Give an example of a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions $\mathbb{C} \to \mathbb{C}$ that converges to some function $f : \mathbb{C} \to \mathbb{C}$ that is not continuous.
(d) Give an example of a sequence $(f_n)_{n\in\mathbb{N}}$ of differentiable functions $f_n : \mathbb{R} \to \mathbb{R}$ that converges to a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} f(x)dx.$$

3. Prove that the series $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ converges locally uniformly on the domain $\{z : Re(z) > 1\}$. What is the derivative $\zeta'(z)$ in terms of a series expansion?

4. Let f be a holomorphic function on $D = \{z : |z| < 1\}$ with $f(0) = 0$. Prove that the series $\phi(z) = \sum_{n=1}^{\infty} f(z^n)$ converges locally uniformly on D.

5. (a) Prove that the power series $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} : \mathbb{C} \to \mathbb{C}$ converges absolutely and locally uniformly on $\mathbb{C}$.

(b) Is the converse of Weierstrass M-test true in general?