Complex Analysis Exercise 10 (Solution)

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1. Let $f : D := \{|z| = 1\} \to \mathbb{C}$ be a continuous function. Assume that there exists a family of polynomials $P_n \to f$ which converges to f uniformly on D. Prove that there exists $F : D \to \mathbb{C}$ which is continuous and holomorphic at interior points such that $F|_D = f$,
Solution. Put $F(z) := \lim_{n\to\infty} P_n(z)$. We show that P_n converge uniformly on $\{|z| \leq 1\}$. In particular, they converge pointwise so F is well-defined. Moreover, this implies that F is continuous and holomorphic in interior as a uniform limit of continuous and holomorphic functions. Clearly, $F|_{|z|=1} = f$. By the maximum principle,

$$\sup_{|z|\leq 1} |P_n(z) - P_m(z)| \leq \max_{|z|=1} |P_n(z) - P_m(z)| \to \infty,\ n, m \to \infty$$

and therefore $P_n(z)$ converge uniformly on the closed disk.

2. Compute Taylor expansion around zero for the following functions:

(a) $z\, cos^2 z$

(b) $sinh(z)$

(c) $\frac{5z-1}{z^2-2z-15}$

(d) $\frac{1}{(z-3)^2}$

(e) $Log \frac{1+iz}{1-iz}$.

Solution.(a) Since $cos\, z = \frac{e^{iz}+e^{-iz}}{2}$, $z\, cos^2 z = z + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1}}{(2k!)} z^{2k+1}$.
(b) $sinh\, z = \frac{e^z - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$.
(c) $\frac{5z-1}{z^2-2z-15} = \frac{2}{z+3} + \frac{3}{z-5} = \sum_{n=0}^{\infty} [(-1)^n \frac{2}{3^{n+1}} - \frac{3}{5^{n+1}}] z^n$.
(d) $\frac{1}{(1-z)^3} = \frac{1}{2} (\frac{1}{1-z})'' = \frac{1}{2} \sum_{n=0}^{\infty} (n+1)(n+2) z^n$ by the local uniform convergence.

(e) In the neighborhood of 1, we have $Log\, u - Log\, v = Log\, \frac{u}{v}$. Since the coefficients of Taylor series only depend on the local behavior of the function,

$$Log\, \frac{1+iz}{1-iz} = Log(1+iz) - Log(1-iz) = 2\sum_{n=0}^{\infty} \frac{i^{2n+1}}{2n+1} z^{2n+1}.$$

3. (a)Calculate Taylor expansion of $\frac{1}{1-z}$ around points $0, i, -1$. Find radius of convergence in each case.
(b) Let $z_0 \in \mathbb{C}^*$. Find the series expansion of $f(z) = \frac{1}{z}$ around z_0 and determine its radius of convergence.
(c) Show that the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$ is equal to e and that the series diverges everywhere on the boundary.
Solution. (a) Around 0, $\frac{1}{1-z} = \sum_{n=1}^{\infty} z^n$. Around i,

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-\frac{z-i}{z-i}} = \frac{1}{1-i} \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^n}.$$

Similarly, around $-i$, $\frac{1}{1-z} = \frac{1}{2}\sum_{n=0}^{\infty} (\frac{z+1}{z})^n$. Note that $\frac{1}{1-z}$ has only one singular point at $z=1$. Therefore the radius of convergences are $1, \sqrt{2}, 2$ respectively.
(b) Similarly, we have

$$\frac{1}{z} = \sum_{n=0}^{\infty} \frac{1}{z_0} (\frac{z_0 - z}{z_0})^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(z_0)^{n+1}} (z - z_0)^n.$$

The convergence of radius is $|z_0|$.
(c) We use Stirling formula $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$. Then the inverse of the radius of convergence is

$$\limsup_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \limsup_{n\to\infty} \sqrt[n]{\sqrt{2\pi n} \frac{(\frac{n}{e})^n}{n^n}} = \frac{1}{e}.$$

4. Consider the series

$$arcsin(z) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{z^{2n+1}}{2n+1}.$$

Determine its radius of convergence and show that $arcsin(z)$ is the unique inverse function of $sin(z)$, wherever it is defined.
Solution. Using the ratio test, the square of convergence of radius is

$$\lim_{n\to\infty} \frac{\left(\frac{(2n)!}{2^{2n}(n!)^2(2n+1)}\right)}{\left(\frac{(2n+2)!}{2^{2n+2}((n+1)!)^2(2n+3)}\right)} = \lim_{n\to\infty} \frac{4(n+1)^2(2n+3)}{(2n+1)^2(2n+2)} = 1.$$

We can easily check that $sin(z)$ is bijective on the domain so the inverse function is also holomorphic. On the other hand the real part of the $arcsin\, z$ defined by a power series coincides with $arcsin\, x$ on the real line hence $arcsin\, z$ is the unique inverse function of $sin\, z$ on the domain.

5. Let $D = \{z||z| < 1\}$ be a unit disk and $\overline{D}$ be the closure. Give an example of a continuous function $f : \overline{D} \to \mathbb{C}$ that is holomorphic on D, but does not have a holomorphic continuation on any domain in $\mathbb{C}$ containing $\overline{D}$.
Solution. We can simply define f as a power series with radius of convergence 1, that converges everywhere on the boundary, for example $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^2}$. f is clearly continuous on $\overline{D}$ and holomorphic on D. To get the last statement, we use the fact that every domain containing $\overline{D}$ contains an open disk with radius $r > 1$ (We can use the fact that $\overline{D}$ is compact).