Complex Analysis Exercise 11(Solution)

Prof. Dr. Paul Biran

Due: 29.11.2019

1. Compute Laurent series for the following functions:

   (a) $\frac{e^{1-z}}{z^2}$ around 0

   (b) $z^4(e^{\frac{1}{z^2}}-1)$ around 0

   (c) $e^{z+\frac{1}{z}}$ in the annulus $0<|z|<\infty$

   (d) $\frac{1}{1-z^2}$ in every annulus around 0 and around 1.

Solution. (a) $\frac{e^{1-z}}{z^3} = e.\frac{1}{z^3}\sum_{n=0}^{\infty}\frac{(-z)^n}{n!} = \sum_{n=-3}^{\infty}\frac{(-1)^{n+1}ez^n}{(n+3)!}$
(b) $z^4(e^{\frac{1}{z^2}}-1) = z^4(\sum_{n=0}^{\infty}\frac{z^{-2n}}{n!}-1) = ...$
(c) $e^{z+\frac{1}{z}} = e^z e^{\frac{1}{z}} = \sum_{n=-\infty}^{\infty}\sum_{j=n}^{\infty}\frac{1}{j!}\frac{1}{(n+j)!}z^n$
(d) Around $z_0=1$: we have two singular points $z=\pm 1$, so the relevant annuli are $0<|z-1|<2$ and $|z-1|>2$. On $0<|z-1|<2$, we have

$$\frac{1}{1-z^2} = -\frac{1}{2}\frac{1}{z-1}+\frac{1}{4}\frac{1}{1+\frac{z-1}{2}} = -\frac{1}{2}\frac{1}{z-1}+\sum_{n=0}^{\infty}\frac{1}{4}(-1)^n\frac{(z-1)^n}{2^n}.$$

On $|z-1|>2$, we have

$$\frac{1}{1-z^2} = -\frac{1}{2}\frac{1}{z-1}+\frac{1}{2(z-1)}\frac{1}{\frac{2}{z-1}+1} = -\frac{1}{2}\frac{1}{z-1}+\frac{1}{2(z-1)}\sum_{n=0}^{\infty}(-2)^n(z-1)^{-n}.$$

Around z_0=0: on $|z|<1$,

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}$$

and on $|z|>1$,

$$\frac{1}{1-z^2} = -\frac{1}{z^2}\sum_{n=0}^{\infty} z^{-2n}.$$

2. Let $f(z) = \frac{z-7}{z^2+z-2}$. Find Laurent expansion for every relevant annulus around zero.

Solution. We have two singular points at $z = 1, -2$. On $|z| < 1$,

$$f(z) = 2\frac{1}{1-z} + \frac{3}{2}\frac{1}{1+\frac{z}{2}} = \sum_{n=0}^{\infty}(2 + \frac{(-1)^n 3}{2^{n+1}})z^n.$$

On $1 < |z| < 2$,

$$f(z) = \frac{3}{2}\frac{1}{1+\frac{z}{2}} - \frac{2}{z}\frac{1}{1-\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{(-1)^n 3}{2^{n+1}}z^n + \sum_{n=0}^{\infty}(-2)\frac{1}{z^{n+1}}.$$

On $|z| > 2$,

$$f(z) = \frac{3}{z}\frac{1}{1+\frac{2}{z}} - \frac{2}{z}\frac{1}{1-z^{-1}} = \frac{3}{z}\sum_{n=0}^{\infty}\frac{(-2)^n}{z^n} - \frac{2}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}.$$

3. (a) Let $f \in \mathrm{Hol}(|z| > 1)$. Assume that f satisfies $\int_{|z|=2} f(z)dz = 0$. Show that f has a primitive in $\{|z| > 1\}$.
(b) Let $f \in \mathrm{Hol}(|z| > R)$ for some $R > 0$ and $f(-z) = f(z)$. Prove that f has a primitive in $\{|z| > R\}$ and $\int_{|z|=r} f dz = 0$, $r > R$.
Solution. (a) f is holomorphic in $|z| > 1$, so there exists a Laurent expansion in this annulus: $f(z) = \sum a_n z^n$. From Laurent theorem and local uniform convergence, we can integrate term by term

$$0 = \int_{|z|=2} f(z)dz = \sum_{-\infty}^{\infty} a_n \int_{|z|=2} z^n dz = a_{-1}2\pi i$$

and get $a_{-1} = 0$. Now we can write a primitive F of f as follows

$$F(z) = \sum_{n \neq -1} \frac{a_n}{n+1} z^{n+1}.$$

One can immediately verify that $F' = f$.
(b) There exists a Laurent expansion $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ in $|z| > R$. We prove $a_{2n+1} = 0$. Let $g := f - \sum_{n=-\infty}^{\infty} a_{2n} z^{2n}$. Then $g(z) = g(-z) = -g(z)$ and hence $g \equiv 0$. By integration by parts, a primitive function of f is $\sum_{-\infty}^{\infty} \frac{a_{2n}}{2n+1} z^{2n+1}$.

4. Prove the following statement: Let f be holomorphic in $a < |z| < b$. Then for every $a < r_1 < r_2 < b$,

$$\int_{|z|=r_1} f dz = \int_{|z|=r_2} f dz.$$

Solution. Expand f into Laurent series in $a < |z| < b$: $f(z) = \sum a_n z^n$. Then for $a < r < b$,

$$\int_{|z|=r} f(z)dz = \sum_{-\infty}^{\infty} a_n \int_{|z|=r} z^n dz = a_{-1} 2\pi i.$$

The result does not depend on r.

5. Find all $\lambda \in \mathbb{C}$ for which there exists entire function f such that $\forall z$, $f(z) = f(\lambda z)$ (f is not constant).
Solution. Any entire function admits Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. By assumption, $f(\lambda z) = \sum_{n=0}^{\infty} \lambda^n a_n z^n = f(z)$. By the uniqueness of Taylor expansion, $\lambda^n a_n = a_n$ for all n. It can happen only when $\lambda^n = 1$ for some n. On the other hand, any n root of unity λ is possible as we can take $f(z) = z^n$.