Complex Analysis Exercise 12 (Solution)

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1. Let A be a square centered at the origin. Denote by s one of sides of A. Let $f : A \to \mathbb{C}$ be holomorphic on interior of A and continuous on the boundary of A such that $f(z) = 0$ for all $z \in s$. Prove that $f = 0$ on A.
Solution. Let $f_1 = f(z), f_2 = f(iz), f_3 = f(-z), f_4 = f(-iz)$. Then each side of A has corresponding f_i which vanishes on that side. Therefore $g(z) := f_1(z). \cdots .f_4(z)$ vanishes on the boundary of A. By the maximum principle, $g \equiv 0$. $\forall z \in A$, out of four points $\{\pm z, \pm iz\} \subset A$, there is at least one zero of f. Let $z_k = 1/k$ be a sequence converging to zero. Denote $w_k \in \{\pm z_k, \pm iz_k\}$ the corresponding zero of f. Then $w_k \to 0$, so we have convergent sequence of zeros of f. Therefore $f = 0$ on A.

2. Let f be an entire function which satisfies the following property: $\forall z \in \mathbb{C}$, there exists $N = N(z)$ (not necessarily constant) such that $f^{(N)}(z) = 0$. Prove that f is a polynomial.
Solution. Denote $Z_n = \{z | f^{(n)}(z) = 0\}$ the set of zeros of $f^{(n)}$. By assumption, $\cup_n Z_n = \mathbb{C}$, so at least one of Z_n should be uncountable. Then we can find a converging sequence in Z_n (We can consider $Z_n \cap \overline{B_r}(0), r \in \mathbb{N}$ for instance and use the fact that $\overline{B_r}(0)$ is compact.). Therefore, $f^{(n)} = 0$ and f is polynomial of degree $\leq n$.

3. Let f be an entire function. Assume that $f(z) \in \mathbb{R}$ for any $z \in (-\frac{1}{2019}, \frac{1}{2019})$. Show that $f(\bar{z}) = \overline{f(z)}$.
Solution. Following Exercise 3, we know that $f^* = \overline{f(\bar{z})}$ is an entire function. $\forall Z \in (-\frac{1}{2019}, \frac{1}{2019})$, $f^*(z) = f(z)$. Since $(-\frac{1}{2019}, \frac{1}{2019})$ is not discrete, $f^*(z) = f(z)$. Therefore, for $w = \bar{z}$, $f(\bar{z}) = f(w) = f^*(w) = \overline{f(z)}$.

4. Let f be holomorphic in the unit disk. Assume that f satisfies $f(\frac{1}{2k}) = f(\frac{1}{2k+1})$, $\forall k \in \mathbb{N}$. Show that f is constant.
Solution. Let $\varphi(z) := \frac{z}{z+1}$ which sends $\frac{1}{k}$ to $\frac{1}{k+1}$. φ is holomorhic in $D = \{|z| < 1\}$, so $f \circ \varphi \in Hol(D)$. We have $f \circ \varphi(\frac{1}{2k}) = f(\frac{1}{2k+1}) = f(\frac{1}{2k})$, so

$f \circ \varphi \equiv f$. In particular, $f(\frac{1}{n+1}) = f \circ \varphi(\frac{1}{n}) = f(\frac{1}{n})$, so f is constant on the convergent sequence $\{1/n\}$. Thus f is constant.

5. Find all singular points for the following functions. Determine their type and compute the residues:

(a) $\frac{1}{z^3 - z^5}$

(b) $\frac{e^z - 1}{z^n}$, $n \geq 1$

(c) $sin\, z \cdot sin\, \frac{1}{z}$

(d) $\frac{1}{e^z + 1}$

(e) $\frac{1}{sin^2(z)}$

(f) $\frac{z}{1 - e^{z^2}}$.

Solution. (a) $\frac{1}{z^3 - z^5} - \frac{1}{z^3(1-z)(1+z)}$. We have simply poles at $z = \pm 1$ and a pole of order 3 at $z = 0$. From the residue formula, $Res_1 = Res_{-1} = -\frac{1}{2}$. Near $z = 0$,

$$f(z) = \frac{1}{z^3}\frac{1}{1 - z^2} = \frac{1}{z^3} + \frac{1}{z} + z + \cdots,$$

so $Res_0 = 1$

(b) $z = 0$ is the singular point. We divide into two cases.

$n = 1$: $\lim_{z \to 0} \frac{e^z - 1}{z} = e^0 = 1$, hence we have a removable singularity.

$n = 2$: $\frac{e^z - 1}{z} = \frac{\sum_{k=1}^{\infty} \frac{z^k}{k!}}{z^n} = \frac{1}{z^{n-1}} + \cdots + \frac{1}{(n-1)!}\frac{1}{z} + \cdots$, so we have pole of order $n - 1$ with $Res_0 = \frac{1}{(n-1)!}$.

(c) We have unique singularity at $z = 0$. We have $\lim_{z \to 0, z \in \mathbb{R}} f(z) = 0$ and $\lim_{z \to 0, z \in i\mathbb{R}} |f(z)| = \infty$, so $\lim_{z \to 0} f(z)$ does not exist. Hence $z = 0$ is essential.

(d) We have singular points at $z = \pi i + 2\pi k i, k \in \mathbb{Z}$. The function is periodic with period $2\pi i$, so all points are of the same type and have the same residue. It is enough to check at $z_0 = \pi i$. z_0 is a simple zero of $e^z + 1$, so it is a simple pole of $\frac{1}{e^z + 1}$. $Res_{z_0} = \lim_{z \to \pi i} \frac{z - \pi i}{e^z + 1} = \lim_{z \to \pi i} \frac{1}{e^z} = -1$.

(e) We have singular points at $z = \pi k, k \in \mathbb{Z}$. Those are zeroes of order 2 of the denominator, so those are poles of order 2. $Res_{\pi k} = \lim_{z \to \pi k} (z - \pi k)^2 \frac{1}{sin^2(z)} = 0$.

(f) We have singular points at $z^2 = 2\pi i k, k \in \mathbb{Z}$. $z = 0$ is the simple zero of numerator and zero of order 2 of denominator, so it is a simple pole. $Res_0 = \lim_{z \to 0} \frac{z^2}{1 - e^z} = \lim_{z \to 0} \frac{2z}{-2ze^{z^2}} = \lim_{z \to 0} -\frac{1}{e^{z^2}} = -1$.

When $z = \sqrt{2\pi i k}$, it is a simple pole. $Res_{\sqrt{2\pi i k}} = \lim_{z \to \sqrt{2\pi i k}} \frac{z(z - \sqrt{2\pi i k})}{1 - e^{z^2}} = \lim_{z \to \sqrt{2\pi i k}} \frac{(z - \sqrt{2\pi i k}) + z}{-2ze^{z^2}} = -\frac{1}{2}$.

6. Let f, g be entire and non-constant functions. Assume that $|f(z)| \leq |g(z)|$ for all z. Prove that there exists $c \in \mathbb{C}$ such that $f(z) = cg(z)$.

Solution. Define $h(z) = \frac{f(z)}{g(z)}$. Since g is not constant, singularities of h are isolated. By assumption, $|h(z)| \leq 1$, $\forall z$ where $g(z) \neq 0$. In particular, $\forall z_0$, $g(z_0) = 0$, h is bounded near z_0. By Riemann extension theorem, z_0 is a removable singularity. We extend h to an entire function . This extension is still bounded since $|h(z_0)| = |\lim_{z \to z_0} h(z)| \leq 1$. Therefore, h is entire and bounded, hence it is a constant function. We conclude $f(z) = cg(z)$, $c \in \mathbb{C}$, $\forall z$ where $g(z) \neq 0$. For z where $g(z) = 0$, $|f(z)| \leq |g(z)| = 0$, so f is also zero and $f(z) = cg(z)$ holds for all $z \in \mathbb{C}$.