Complex Analysis Exercise 2 (Solution)

Prof. Dr. Paul Biran

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1. Let A be an open and path-connected subset in $\mathbb{C}$. If we write

$$A = U \cup V$$

where U and V are open subsets and $U \cap V = \emptyset$, then prove either $U = A$, $V = \emptyset$ or $U = \emptyset$, $V = A$.

Solution. Proof by contradiction. If $A = \emptyset$, then the statement is trivial, so let $A \neq \emptyset$. Suppose neither $U = A$, $V = \emptyset$ or $U = \emptyset$, $V = A$ holds. Choose two points $u \in U$, $v \in V$ and take a continuous path $\gamma : [0, 1] \to A$ such that $\gamma(0) = u$ and $\gamma(1) = v$. Take

$$t_0 := \inf\{t \in [0, 1] : \gamma(t) \in V\}$$

Since U and V are nonempty open subset, $0 < t_0 < 1$. In fact, we can also see $\gamma(t_0) \in U$ and $\gamma(t_0) \in V$. Which leads to a contradiction. $\square$

2. (a) Prove (without using the Cauchy-Riemann equation) that functions

$$f(z) = Re(z), \; g(z) = Im(z)$$

are not differentiable at any point.

(b) Let $a, b \in \mathbb{C}$. Find all points in $\mathbb{C}$ where $af(z) + bg(z)$ is differentiable.

Solution. (a) We have

$$\lim_{h \to 0, h \in \mathbb{R}} \frac{f(z+h) - f(z)}{h} = 1.$$

On the other hand, we have

$$\lim_{h \to 0, h \in i\mathbb{R}} \frac{f(z+h) - f(z)}{h} = 0.$$

Therefore the general limit does not exist. One can prove the statement for $g(z)$ with a similar argument.

(b) We can rewrite the expression as

$$af(z) + bg(z) = a\frac{z+\overline{z}}{2} + b\frac{z-\overline{z}}{2i} = z(\frac{a}{2} - \frac{ib}{2}) + \overline{z}(\frac{a}{2} + \frac{ib}{2})$$

The function $af(z)+bg(z)$ is differentiable if and only if $\overline{z}(\frac{a}{2}+\frac{ib}{2})$ is diffentiable. From (a), we conclude that the function is differentiable if and only if $a = -ib$.
□

3. Find at which points derivatives of the following functions exists. Compute these derivatives.

(a) $f(z) = \frac{3z^2+2z}{z^4-1}$

(b) $f(z) = e^{\overline{z}}$

(c) $f(z) = z(z + \overline{z}^2)$.

Solution. (a) The function is defined when $z^4 \neq 1$. Then it is differentiable since it is a rational function. From the chain rule, we have

$$f'(z) = \frac{(6z+2)(z^4-1) - (3z^2+2z)4z^3}{(z^4-1)^2}.$$

(b) Let $z = x + iy$. Then,

$$f(z) = e^{x-iy} = e^x cos(y) - ie^x sin(y) = u + iv.$$

From the Cauchy-Riemann equations, we should have

$$u_x = e^x cos(y) = v_y = -e^x cos(y)$$

and

$$u_y = -e^x sin(y) = -v_x = e^x sin(y).$$

Since $e^x \neq 0$, the derivative exists if and only if $cos(y) = sin(y)$. Therefore the derivative does not exists at any point.
(c) The function is differentiable if and only if $z\overline{z}^2$ is differentiable. Let $g(z) = z\overline{z}^2$ and we separate the problem into two cases.

If $z = 0$, $g'(z) = \lim_{z\to 0} \frac{g(z)-g(0)}{z} = \lim_{z\to 0} \overline{z}^2 = 0$. Hence the function is differentiable and $f'(0) = 0$.

If $z \neq 0$, f is differentiable if and only if $\frac{f(z)}{z}$ is differentiable. By checking the Cauchy-Riemann equation, we know that f is not differentiable when $z \neq 0$.
□

4. Prove

(a) $cos(\frac{\pi}{2} - z) = sin(z)$

(b) $cos(z) = cosh(iz)$.

Solution. (a) We have

$$cos(\frac{\pi}{2} - z) = \frac{e^{(\frac{\pi}{2}-z)i} + e^{-(\frac{\pi}{2}-z)i}}{2} = \frac{ie^{-zi} - ie^{zi}}{2} = sin(z).$$

(b) We have

$$cos(z) = \frac{e^{iz} + e^{-iz}}{2} = cosh(iz).$$

□