Complex Analysis Exercise 4 (Solution)

Prof. Dr. Paul Biran

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1. Prove or give a counterexample of the following statements:

   (a) $Log(zw) = Log(z) + Log(w)$

   (b) $Log(z^{-1}) = -Log(z)$, $z \neq 0$

   (c) $z^{w+1} = z \cdot z^w$.

Solution.

   (a) False. Take $z = w = e^{\frac{3\pi}{4}i}$.

   (b) False. For $z = re^{i\theta} \notin \mathbb{R}^-$, $Log(z) = lnr + i\theta$ but $Log(z^{-1}) = -lnr - i\theta$.

   (c) True. $z^{w+1} = e^{(w+1)Logz+(w+1)2\pi ik} = e^{w+1Logz+w2\pi ik} = z \cdot z^w$.

2. Solve the equation $cos(z) = 0$, $z \in \mathbb{C}$.

Solution. This question is similar to Problem 7 in the last exercise sheet. By the similar argument, we can easily get $z = \frac{(2n+1)\pi}{2}$.

3. Calculate the integral $\int_\gamma Rez + Imzdz$ where $\gamma(t)$ is given by:

   (a) $t + it^2$, $t \in [-1, 1]$

   (b) $1 + t + i(2 + t)$, $t \in [0, 1]$

   (c) e^{it}, $t \in [0, 2\pi]$.

Solution.

   (a) $\int_\gamma Rez + Imzdz = \int_{-1}^1 (t + t^2)(1 + 2it)dt = \frac{2}{3} + \frac{4}{3}i$.

   (b) $\int_\gamma Rez + Imzdz = \int_0^1 (1 + t + 2 + t)(1 + i)dt = 4 + 4i$.

   (c) $\int_\gamma Rez + Imzdz = \int_0^{2\pi} (cost + sint)(-sint + icost)dt = \int_0^{2\pi} (icos^2 t - sin^2 t)dt = i\pi - \pi$.

4. Compute the following path integrals:

(a) $\int_\gamma cos(Rez)dz$ where γ is a circle around i with radius 1 with counter-clockwise orientation.

(b) $\int_\gamma \frac{Logz}{z}dz$ $\gamma(t) = e^{it}$, $t \in [0, \pi]$.

(c) $\int_\gamma (\overline{z})^n dz$ for any $n \in \mathbb{Z}$ and γ where γ is the unit circle with counter-clockwise orientation.

Solution.

(a) $\int_\gamma cos(Rez)dz = \int_0^{2\pi} cos(cos\theta)i(cos\theta + isin\theta)d\theta = \int_{-\pi}^{\pi} cos(cos\theta)sin\theta d\theta + i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos(cos\theta)cos\theta d\theta + i\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} cos(cos\theta)cos\theta d\theta = 0.$

(b) $Log(e^{it}) = it$ when $0 \le t < \pi$, so

$$\int_\gamma \frac{Logz}{z}dz = \int_0^\pi \frac{it}{e^{it}}ie^{it}dt = \int_0^\pi -tdt = -\frac{\pi^2}{2}.$$

(c) On the unit circle, $\overline{z} = z^{-1}$. Thus, the integral is 0 if $n \ne 1$ and $2\pi i$ if $n = 1$.

5. For any integer $n \ge 1$, prove

$$\int_0^{2\pi} cos^{2n}tdt = 2^{1-2n}\binom{2n}{n}\pi.$$

Solution. Let γ be a curve which parametrizes the unit circle with counter-clockwise orientation. We have

$$\int_\gamma z^{-1}(z+z^{-1})^{2n}dz = \int_\gamma \binom{2n}{n}z^{-1}dz + \int_\gamma (\text{other powers of} z)dz = 2\binom{2n}{n}\pi i.$$

On the other hand, we have

$$\int_0^{2\pi} e^{-it}[e^{it} + e^{-it}]^{2n}ie^{it}dt = \int_0^{2\pi} 2^{2n}(cost)^{2n}idt.$$

The desired equality follows immediately.

6. Give an example of a (continuous) path $\gamma : [0, 1] \to \mathbb{C}$ of infinite length.
Solution. Let

$$\gamma(t) = \begin{cases} t + it \cdot sin(t^{-1}) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Consider the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{2}{(2n+1)\pi}$. Then,

$$|\gamma(a_{n+1}) - \gamma(a_n)| > \frac{2}{(2n+3)\pi} + \frac{2}{(2n+1)\pi} > \frac{4}{(2n+3)\pi}.$$

So,

$$\text{Length}(\gamma) \geq \sum_{n=0}^{\infty} |\gamma(a_{n+1}) - \gamma(a_n)| \geq \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+3} = \infty.$$

8. Let $D \subset \mathbb{C}$ be a domain. Let $\gamma : [a, b] \to D$ be a piecewise regular path such that $f \circ \gamma$ is continuous. Prove that

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\dot{\gamma}(t)dt. \tag{1}$$

Solution. Step 1. We prove the following lemma:

Lemma 1. *Let $g, h : [a, b] \to \mathbb{R}$ be continuous functions. Let $\Delta = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$ and $T = \{\tau_0, \cdots, \tau_{n-1}\}$, $\Sigma = \{\sigma_0, \cdots, \sigma_{n-1}\}$ be two choice of intermediate points: $\tau_i, t_i \in [t_i, t_{i+1}]$. Define*

$$S(f, g; \Delta, T, \Sigma) := \sum_{j=0}^{n-1} f(\tau_j)g(\sigma_j)(t_{j+1} - t_j).$$

Then,

$$\lim_{l(\Delta)\to 0} S(f, g; \Delta, T, \Sigma) = \int_a^b g(t)h(t)dt,$$

where $l(\Delta) = max_{0\leq j\leq n-1}|t_{j+1} - t_j|$.

Proof. The proof is a standard argument involving the intermediate value theorem and the definition of the Riemann integral. □

Step 2. We prove another lemma

Lemma 2. *Let f, γ be as above in (1). Assume that γ is regular on $[a, b]$. Then (1) holds.*

Proof. Let $\Delta = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$ and $T = \{\tau_0, \cdots, \tau_{n-1}\}$ be a choice of intermediate points. By Lagrange theorem, there exists $s_j, r_j \in [t_j, t_{j+1}]$ such that $x(t_{j+1}) - x(t_j) = \dot{x}(s_j)(t_{j+1} - t_j)$ and

$y(t_{j+1}) - y(t_j) = \dot{y}(r_j)(t_{j+1} - t_j)$. Then,

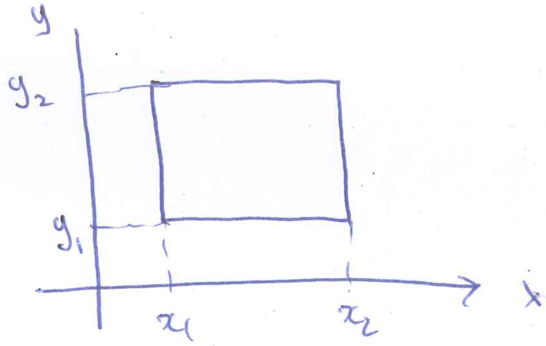
$$\begin{aligned} S(f;\Delta,T) &:= \sum_{j=0}^{n-1} f(\gamma(\tau_j))(\gamma(\tau_{j+1}) - \gamma(\tau_j)) \\ &= \sum_{j=0}^{n-1} u(x,y)\dot{x}(s_j)(t_{j+1} - t_j) - v(x,y)\dot{y}(r_j)(t_{j+1} - t_j) \\ &\qquad + i\sum_{j=0}^{n-1} u(x,y)\dot{y}(r_j)(t_{j+1} - t_j) + v(x,y)\dot{x}(s_j)(t_{j+1} - t_j). \end{aligned}$$

Using Step 1, it is easy to see that

$$\lim_{l(\Delta)\to 0} S(f;\Delta,T) = \int_a^b f(\gamma(t))\dot{\gamma}(t)dt.$$

In case the curve is only piecewise regular, then there exists a partition where the curve is regular on each interval. We can use the additivity of the integral to get the result. □

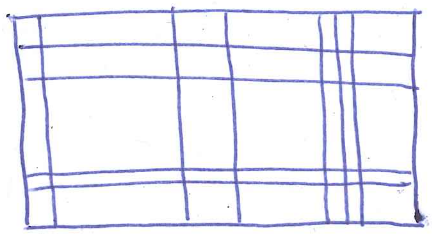
7. For a rectangle ABCD as below, define $I_{ABCD} := \int_{x_1}^{x_2} e^{2\pi i x} dx \int_{y_1}^{y_2} e^{2\pi i y} dy$



Note that $\int_{x_1}^{x_2} e^{2\pi i x} dx = 0 \iff x_2 - x_1 \in \mathbb{Z}$. Therefore, $I_{ABCD} = 0$ iff at least one of the sides of ABCD is of integer length.

<u>Claim</u> I_{ABCD} is an additive function. Namely, if a given rectangle ABCD is divided into smaller rectangles $A_iB_iC_iD_i$, then $I_{ABCD} = \sum I_{A_iB_iC_iD_i}$

Pf) First, observe that it is enough to show the claim for "grid like" decompositions:



Then, by the linearity of the integral, we can easily check the additivity □

Using the above Claim,

$$I_{MNPQ} = \sum_i \underbrace{I_{\text{smaller piece}}}_{=0 \text{ by assumption}} = 0.$$

Therefore MNPQ has at least one side of integer length.