Complex Analysis Exercise 5

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1. Compute the following integrals:
   (a) $\int_{|z|=3} \frac{z}{(z-1)(z-i)} dz$,
   (b) $\int_{|z|=2} \frac{e^z}{z^2-1} dz$,
   (c) $\int_\gamma 2z - 3\overline{z} + 1 dz$ where γ is the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$,
   (d) $\int_\gamma \frac{dz}{\sqrt{z}}$ $(\sqrt{z} = e^{\frac{1}{2}Log z})$, where $\gamma = \{e^{it} | 0 \leq t \leq \pi\}$.

Solution. (a) We have

$$\frac{z}{(z-1)(z-i)} = \frac{1}{1-i}\left(\frac{z}{z-1} - \frac{z}{z-i}\right).$$

By Cauchy's integral formula,

$$\int_{|z|=3} \frac{z}{(z-1)(z-i)} dz = \frac{1}{1-i}(2\pi i.1 - 2\pi i.i) = 2\pi i.$$

(b) We have

$$\int_{|z|=2} \frac{e^z}{z^2-1} dz = \frac{1}{2}\int_{|z|=2} \frac{e^z dz}{z-1} - \frac{1}{2}\int_{|z|=2} \frac{e^z dz}{z+1} = \pi i(e - e^{-1}).$$

(c) $2z+1$ has a primitive, so $\int_\gamma (2z+1)dz = 0$ for a closed curve γ. Hence,

$$\int_\gamma 2z - 3\overline{z} + 1 dz = \int_\gamma -3\overline{z} dz = -3\int_0^{2\pi} (3cost - 2isint)(-3sint + 2icost)dt = -36\pi i.$$

(d) The function has a primitive:

$$(2\sqrt{z})' = \frac{d}{dz}(2e^{\frac{1}{2}Log z})' = \frac{1}{\sqrt{z}}.$$

in $\mathbb{C} - \mathbb{R}^-$. Hence,

$$\int_\gamma \frac{dz}{\sqrt{z}} = 2e^{\frac{1}{2}\pi i} - 2e^0 = 2i - 2.$$

2. (a) Show that for any rational function $R(x)$ which is defined on $[-1,1]$, the following holds:

$$\int_0^{2\pi} R(cos\theta)d\theta = \int_{|z|=1} R(\frac{1}{2}(z+z^{-1}))\frac{dz}{iz}$$

(b) Compute $\int_0^{2\pi} \frac{d\theta}{a+cos\theta}$ for $a > 1$.
Solution. (a) Let $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then,

$$\int_{|z|=1} R(\frac{1}{2}(z+z^{-1}))\frac{dz}{iz} = \int_0^{2\pi} R\left(\frac{e^{it}+e^{-it}}{2}\right)\frac{ie^{it}dt}{ie^{it}} = \int_0^{2\pi} R(cos\theta)d\theta.$$

(b) Roots of $z^2+2az+1$ are $z = -a \pm \sqrt{a^2-1}$. Let $z_1 = -a+\sqrt{a^2-1}$ and $z_2 = -a-\sqrt{a^2-1} < -1$. Then,

$$\int_0^{2\pi} \frac{d\theta}{a+cos\theta} = \int_{|z|=1} \frac{a}{a+\frac{1}{2}(z+z^{-1})}\frac{dz}{iz} = \frac{2}{i}\int_{|z|=1} \frac{1}{z-z_2}\frac{dz}{(z-z_1)}.$$

By Cauchy's formula, this is equal to $4\pi\frac{1}{z_1-z_2} = \frac{2\pi}{\sqrt{a^2-1}}$.

3. Compute $\int_0^t xsin(2x)dx$ using the complex integral $\int_{[0,t]} ze^{2iz}dz$.
Solution. Let $\gamma(x) = x + 0.i$, $0 \le x \le t$.Then,

$$\int_\gamma ze^{2iz}dz = \int_0^t xe^{2ix}dx.$$

Using integration by parts, we get

$$(\frac{cos2t}{4} + \frac{tsin2t}{2} - \frac{1}{4}) + i(\frac{sin2t}{4} - \frac{tcos2t}{2}).$$

The imaginary part is the desired quantity.

4. (a) Liouville's theorem says that every bounded entire function is constant. Prove the theorem in the following way: Let f be an entire and bounded function. Pick $a \neq b \in \mathbb{C}$. Let R be a real number greater than $|a|$ and $|b|$. Calculate $\int_{|z|=R} \frac{f(z)dz}{(z-a)(z-b)}$ and check what happens when $R \to \infty$.
(b) Let f be an entire function with two periods. Show that f is constant.
(c) Prove the fundamental theorem of algebra: Let p be a nonconstant polynomial with complex coefficients. Then there exists $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.
(d)Find all entire function f such that $|f'(z)| < e^{-(Rez)^2}$ for all $z \in \mathbb{C}$.
Solution.(a) Assume that $|f(z)| \le C$ for all $z \in \mathbb{C}$. Pick $R > max(|a|,|b|)$. By Cauchy's integral formula,

$$\int_{|z|=R} \frac{f(z)dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b}(f(a)-f(b)).$$

On the other hand,

$$\left|\int_{|z|=R}\frac{f(z)dz}{(z-a)(z-b)}\right|\le 2\pi R\frac{C}{(R-|a|)(R-|b|)}.$$

Therefore,

$$\frac{2\pi}{|a-b|}|f(a)-f(b)|\to 0,\ \ R\to\infty,$$

and hence f is constant.
(b) We may assume two periods to be 1 and $\tau\in\mathbb{C}-\mathbb{R}$. Let Δ be a parallelogram spanned by $1,\tau$ and let $M=max\{|f(z)|,z\in\Delta\}$. By the periodicity, $|f(z)|\le M$. By (a), f is constant.
(c) Let $p(z)=a_nz^n+\cdots+a_0$ where $n\ge 0$ and $a_n\ne 0$. As $\frac{p(z)}{z^n}\to a_n$ as $z\to\infty$, there exists $R>0$ such that

$$|\frac{p(z)}{z^n}|>\frac{|a_n|}{2},\ \ |z|>R.$$

Suppose that $p(z)=0$ has no roots so that p^{-1} is holomorphic. Let γ_r be a clockwise oriented path of radius r around the origin. By Cauchy's integral formula,

$$0\ne|p(0)^{-1}|=\left|\frac{1}{2\pi i}\int_{\gamma_r}\frac{dz}{zp(z)}\right|\le\frac{2}{|a_n|r^n}\to 0,\ \ r\to\infty.$$

which leads to a contradiction.
(d) If f is an entire function, f' is also entire. Since

$$|f'(z)|<e^{-(Rez)^2}\le 1,$$

f' is a constant function. On the other hand,

$$c=|f'(z)|<e^{-(Rez)^2}\to 0,\ \ Rez\to\infty,$$

so $f'=0$ and f is a constant function.

5. Let $D\subset\mathbb{C}$ be a unit disk at the origin. Find all functions $f(z)$ which are holomorphic on D and which satisfy

$$f(\frac{1}{n})=n^2f(\frac{1}{n})^3,\ \ n=2,3,4,\cdots$$

Solution. We rewrite this as

$$f(1/n)(f(1/n)-1/n)(f(1/n)+1/n)=0.$$

At each n, one of the following holds: $f(1/n)=0$, $f(1/n)=1/n$, or $f(1/n)=-1/n$. At least one of these three equations must hold for infinitely many n. From the lemma below, either $f(z)=0$, $f(z)=z$, or $f(z)=-z$.

Lemma 1. *Let f be a holomorphic function on a (connected) domain D. Then the following statements are equivalent:*

(a) $f(z) = 0$ *for all* $z \in D$.

(b) The set $Z = \{z \in D | f(z) = 0\}$ *has a limit point in* D.

Proof. $(a) \Rightarrow (b)$: Clear.
$(b) \Rightarrow (a)$: Let a be a limit point of Z in D. Take $\epsilon > 0$ such that $B_\epsilon(a) \subset D$ and let $f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$ be the Taylor expansion of f centered about a. Suppose that not all a_k are zero and let K be the smallest k such that $a_K \neq 0$. Then

$$f(z) = (z-a)^K g(z), \text{ where} g(z) = \sum_{k=K}^{\infty} a_k (z-a)^{k-K}.$$

There exists $\delta > 0$ such that $g(z) \neq 0$ on $B_\delta(a)$. Thus on $B_\delta(a)$, $f(z) = 0$. only holds at a, contracting the fact that a is a limit point of Z.Hence $f(z) = 0$ on $B_\epsilon(a)$. To show $Z = D$, use the connectivity of D. $\square$