Complex Analysis Exercise 6 (Solution)

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1. Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise C^1 function. For $z \notin \gamma$, let $\lambda_z(t) = \gamma(t) - z$. Show that there exists a C^1 function $\theta_z(t) : [a,b] \to \mathbb{R}$ such that $\theta_z(t) \in arg(\lambda_z(t))$.

Solution. We start from the following lemma.

Lemma 1. *Let $-\pi < \theta \leq \pi$ and let $R_\theta = \{re^{i\theta} | r \geq 0\} \subset \mathbb{C}$ be the ray with argument θ. Then there exists a continuous function $\phi : \mathbb{C} \setminus R_\theta \to \mathbb{R}$ such that for all $z \in \mathbb{C} \setminus R_\theta$, $\phi(z) \in arg(z)$.*

Proof. Let

$$v(x+iy) = \begin{cases} arcsin(\frac{y}{\sqrt{x^2+y^2}}) & \text{if } x \geq 0, \\ \pi + arcsin(\frac{y}{\sqrt{x^2+y^2}}) & \text{if } x < 0 \text{ and } y > 0, \\ -\pi - arcsin(\frac{y}{\sqrt{x^2+y^2}}) & \text{if } x < 0 \text{ and } y < 0. \end{cases}$$

It is straightforward to check that v is a C^1 choice of arguement on $\mathbb{C} \setminus R_\pi$. For arbitrary θ, choose a rotation to get a C^1 choice of argument. □

By translation, we may assume $z \neq 0$. Let $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$ be a partition of $[a,b]$ such that for each i, the image $\gamma([t_i, t_{i+1}])$ is contained in an open half plane (One can check such partition always exists.). For each i, choose $-\pi < \theta_i \leq \pi$ such that $\gamma([t_i, t_{i+1}]) \subset \mathbb{C} \setminus R_{\theta_i}$. Let ϕ_i be a continuous choice of argument on $\mathbb{C} \setminus R_{\theta_i}$ from the above lemma. Denote $\delta_i = \phi(\gamma(t_{i+1})) - \phi_i(\gamma(t_i))$ and

$$\theta : [a,b] \to \mathbb{R}, \ [t_i, t_{i+1}] \ni t \mapsto \phi_0(\gamma(t_0)) + \sum_{k=0}^{i-1} \delta_k + \phi_i(\gamma(t)) - \phi_i(\gamma(t_i)).$$

It is straight forward to check that $\theta(t)$ is C^1 and $\theta(t) \in arg(\gamma(t))$.

2. Let $\gamma : [a,b] \to \mathbb{C}$ be a closed path and let $\theta_z(t)$ be a function as above. In class, we defined the winding function

$$I_\gamma : \mathbb{C} \setminus \gamma \to \mathbb{Z}, \ I_\gamma(z) = \frac{\theta_z(b) - \theta_z(a)}{2\pi}.$$

Prove the following basic properties of the winding function $I_\gamma(z)$:

(a) $I_\gamma(z) = -I_{-\gamma}(z)$.

(b) Let $\sigma : [\alpha, \beta] \to [a, b]$ be a C^1 function with $\sigma(\alpha) = a$ and $\sigma(\beta) = b$. Then, $I_{\gamma \circ \sigma}(z) = I_\gamma(z)$. Therefore the winding number does not change under reparametrization.

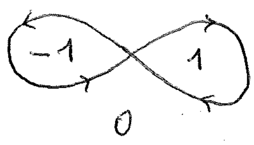
Solution. (a) From the definition, we have

$$I_{-\gamma}(z) = \frac{\theta_z(a) - \theta_z(b)}{2\pi} = -I_\gamma(z).$$

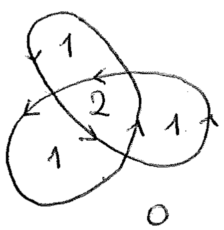
(b) This is also from the definition. We can also see this from the fact that the path integral is indepent of the choise of parametrization.

3. Find the winding number $I_\gamma(z)$ for all $z \in \mathbb{C} \setminus \gamma$ where γ is:

(a)



(b)



Solution. Numbers written as above.

4. Give examples of closed paths γ, such that

(a) $I_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus \gamma$, but γ is non-constant,

(b) the complement of γ has exactly 4 connected components with winding numbers $-1, 0, 1$ and 3,

(c) for every $k \in \mathbb{Z}$, there is $z_k \in \mathbb{C} \setminus \gamma$ with $I_\gamma(z_k) = k$.

Solution. (a) Take any closed path δ and let $\gamma = \delta + (-\delta)$ be the path that runs through δ twist, but with opposite orientation for the second time.

(b) Let γ_1, γ_2, and γ_3 be disjoint loops with starting point and end point in $0 \in \mathbb{C}$(i.e. they intersect only at 0 and no path is contained in the interior of another path). Let γ be a path that runs once through γ_1 with negative orientation, once through γ_2 with positive orientation and three times throught γ_3 with positive orientation.
(c) For each $k \geq 1$, define

$$\gamma_k : [\frac{k-1}{k}, \frac{k}{k+1}] \to \mathbb{C}, \ t \mapsto \frac{i}{k} + \frac{1}{k} e^{i(2\pi k(k+1)(t-\frac{k-1}{k})-\frac{\pi}{2})},$$

i.e. γ_k runs through a circle of radius k^{-1} around ik^{-1} with positive orientation starting from $0 \in \mathbb{C}$. Define

$$\gamma_+ = \sum_{k=1}^{\infty} \gamma_k : [0,1] \to \mathbb{C}$$

to be the composition of all γ_k extended to 1 by $\gamma_{\geq 1}(1) = 0$. Let

$$\gamma_- : [-1, 0] \to \mathbb{C}, \ t \mapsto -\gamma_+(-t).$$

and let $\gamma = \gamma_+ + \gamma_1 : [-1, 1] \to \mathbb{C}$. For $k \geq 1$, set $z_k = \frac{i}{k} + \frac{i}{k+1}$, for $k \leq -1$, set $z_k = \frac{i}{k} + \frac{i}{k-1}$ and set $z_0 = 1$.

5. Let $P \in \mathbb{C}[z]$ be a polynomial and let γ be a simple closed path with positive orientation. Assume that no root of $P(z) = 0$ lies on γ. Prove:

(a) $\frac{1}{2\pi i} \int_\gamma \frac{P'(z)}{P(z)} dz$ is the number of roots of P which are encircled[1] by γ (taking multiplicity into account),

(b) $\frac{1}{2\pi i} \int_\gamma \frac{zP'(z)}{P(z)} dz$ is the sum of the roots of P encircled by γ (with multiplicity).

Solution. (a) Using the fundamental theorem of algebra, we can write

$$P(z) = \alpha \prod_{j=1}^{k} (z - z_j)^{r_j},$$

where z_j are different roots of P with multiplicities r_j. Then

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{k} \frac{r_j}{z - z_j},$$

so

$$\frac{1}{2\pi i} \int_\gamma \frac{P'}{P} dz = \sum_{j=1}^{k} r_j I_\gamma(z_j),$$

[1]This is actually a nontrivial notion. Jordan curve theorem assures that we can define this notion rigorously.

which is the number of roots of P which are encircled by γ

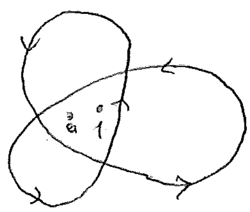
(b) In a similar way,

$$\frac{1}{2\pi i}\int_\gamma \frac{zP'}{P}dz = \frac{1}{2\pi i}\sum_{j=1}^{k} r_j \int_\gamma \frac{z}{z-z_j}dz = \sum_{j=1}^{k} r_j z_j I_\gamma(z_j),$$

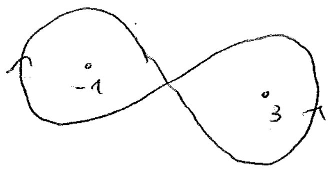
which is the sum of the roots of P encircled by γ.

6. Compute the following integrals:

(a) $\int_\gamma \frac{e^z}{z(z-1)}dz$ where γ is the following path:



(b) $\int_\gamma \frac{z^2+2}{(z-3)(z+1)}dz$ where γ is the following path:



Solution. (a) We have

$$\int_\gamma \frac{e^z}{z(z-1)}dz = \int_\gamma \frac{e^z}{z-1}dz - \int_\gamma \frac{e^z}{z}dz = 2\pi i I_\gamma(1)e - 2\pi i I_\gamma(0) = 2e-2.$$

(b) We have

$$\int_\gamma \frac{z^2+2}{(z-3)(z+1)}dz = \frac{1}{4}\int_\gamma \frac{z^2+2}{z-3}dz - \frac{1}{4}\int_\gamma \frac{z^2+2}{z+1}dz = 7\pi i.$$