Complex Analysis Exercise 7 (Solution)

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1. Let k be a field and $p(x)$ be a nonconstant polynomial with coefficients in k. If $p(x_0) = 0$ for some $x_0 \in k$, show that $p(x) = (x - x_0)q(x)$ for some polynomial $q(x)$.
Solution. We do by induction on n. When $n = 1$, the statement is clear. We assume that the statement holds when the degree is less than equal to $n - 1$. Write

$$p(z) = a_n z^n + \cdots + a_0$$

and let $\tilde{p}(z) = p(z) - a_n(z - z_0)^n$. The degree of $\tilde{p}(z)$ is at most $n - 1$. By induction hypothesis, $\tilde{p}(z) - p(z_0) = (z - z_0)\tilde{q}(z)$ for some polynomial $\tilde{q}(z)$. Substituting $\tilde{p}(z)$ back, we get the statement in degree n.

2. Use Cauchy estimates to prove the following statement: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that there exsits $n \in \mathbb{N}$ and $R, C > 0$ such that

$$|f(z)| < C|z|^n, \text{ for any } |z| > R.$$

Then f is a polynomial in z whose degree is less than or equal to n.
Solution. Pick $k = n + 1$. For any $z \in \mathbb{C}$ and any $r > maz(|z|, R)$,

$$|f^{(k)}(z)| \leq \frac{k!}{2\pi} Cr^n \frac{1}{(f - |z|)^{k+1}} 2\pi r = Ck! \frac{r^{n+1}}{(r - |z|)^{n-2}} \longrightarrow 0, \ r \to \infty.$$

Therefore $f^{(n+1)} = 0$ and f is a polynomial with degree at most n (for the last statement, we can prove, for instance, by the induction on n).

3. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.
Solution. Suppose $f(\mathbb{C})$ is not dense in $\mathbb{C}$, i.e. there exists $z_0 \in \mathbb{C}$ and $\epsilon > 0$ such that $B_\epsilon(z_0) \cap f(\mathbb{C}) =$. Then $|f(z) - z_0| \geq \epsilon$ for all $z \in \mathbb{C}$ and hence the function

$$g(z) = \frac{1}{f(z) - z_0}$$

is holomorphic and bounded. By Liouville theorem, g is constant and f is constant.

4. Prove that there is not entire function f such that $\forall z \in \mathbb{C}$, $|f(z)| > |z|$.
Solution. Since $|f(z)| > |z|$, the function $g(z) = f(z)^{-1}$ is an entire function. Since $g(z)$ is bounded, $g(z)$ is constant. Therefore f is constant.

5. Let f be an entire function. Prove that in each of the following cases, f is constant:

(a) f satisfies $Im(f(z)) \leq 0$ for all $z \in \mathbb{C}$

(b) $|f(z)| \neq 1$ for all $z \in \mathbb{C}$

(c) f does not receive any value in $\mathbb{R}^- = \{x \in \mathbb{R} | x \leq 0\}$.

Solution. (a) Let $g(z) = f(z) - i$. Since $Img \leq -1$, $g(z)^{-1}$ is entire and bounded. By Liouville theorem, h is constant and hence f is constant.
(b) First of all we claim that either $|f(z)| \leq 1$, $\forall z \in \mathbb{C}$ or $|f(z)| \geq 1$, $\forall z \in \mathbb{C}$. Assume by contradiction that this is not true. Namely, there exist z, w such that $|f(z)| > 1$ and $|f(z)| < 1$. Choose a path $\gamma : [0, 1] \to \mathbb{C}$ which connects the two points. Let $\phi(t) = |f(\gamma(t))|$. By intermediate value theorem, there exists t such that $\phi(t) = 1$ contradiction.

For both two cases, we can use Liouville theorem to deduce f is constant.
(c) There is a holomorphic function $\phi : \mathbb{C} \setminus \mathbb{R}^- \to \mathbb{C}$ such that

$$Im\phi \subset \{z | Rez > 0\}.$$

(e.g. take a branch of $\sqrt{z}$ and $\phi(z) = e^{\frac{1}{2}Logz}$) Then $g = \phi \circ f$ is an entire function and $Re(g) > 0$. Therefore, g is a constant function and f is a constant function.

6. Let $\mathbb{D} = \{z \in \mathbb{C} || z| < 1\}$ be a unit disk. Let f be a holomorphic function on the unit disk. Assume that $|f(z)| \leq |f(z^2)|$ in $\mathcal{D}$. Prove that f is constant.
Solution. On the unit disk, $|z^2| < |z|$. For $0 < r < 1$, $\max\{|f(z)|||z|leqr\}$ is attained inside the disk $\{|z| < r\}$. Therefore $f(z)$ is constain in the disk $\{|z| \leq r\}$. As we can take any $r < 1$, f is constant on the unit disk.

7. Let $\mathbb{D}$ be the open disk as above. Find all biholomorphic function $f : \mathbb{D} \to \mathbb{D}$.
Solution. We fist prove two lemmas.

Lemma 1. *Any biholomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ can be written as a rotation.*

Proof. By the Schwarz lemma, $|f(z)| \leq |z|$ and on the other hand, again by the Schwarz lemma, $|z| = |f^{-1}(f(z))| \leq |f(z)|$. Therefore, $|f(z)| = |z|$ and once more by the Schwarz lemma, f is a rotation $z \mapsto e^{i\theta} z$. □

Lemma 2. *Let $w \in \mathbb{D}$. There exists a biholomorphic function $\psi : \mathbb{D} \to \mathbb{D}$ with $\psi(0) = w$.*

Proof. Let $\psi = \frac{z+w}{\bar{w}z+1}$. It is easy to check that $\psi(\mathbb{D}) \subset \mathbb{D}$ with the inverse $\psi^{-1} = \frac{z-w}{1-\bar{w}z}$. □

Let $f : \mathbb{D} \to \mathbb{D}$ be a biholomorphic function and $w = f(0)$ and let ψ be the biholomorphic map from Lemma 2. Then $f^{-1} \circ \psi$ is a biholomorphic map fixing 0. By Lemma 1, we deduce that

$$f(z) = e^{i\theta} \frac{z+w}{\bar{w}z+1}$$

for $w \in \mathbb{D}$ and $-\pi < \theta \leq \pi$.