Complex Analysis Exercise 8 (Solution)

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Due: 08.11.2019

1. Let $Log(z) = ln|z| + iArg(z)$ with $-\pi < Arg(z) \leq \pi$ denote the main branch of the complex logarithm. For which z, w and $k \in \mathbb{Z}$ do the following identities hold?

(a) $Log(z.w) = Log(z) + Log(w)$,

(b) $Log(z^k) = kLog(z)$,

(c) $exp(Log(z)) = z$,

(d) $Log(exp(z)) = z$,

(e) $Log'(z) = \frac{1}{z}$.

Solution. (a) $Log(z.w) = ln|z| + ln|w| + iArg(z.w)$, so we have $Log(z.w) = Log(z) + Log(w)$ if and only if $-\pi < Arg(z) + Arg(w) \leq \pi$.
(b) $Log(z^k) = kLog(z)$ if and only if $Arg(z^k) = kArg(z)$, i.e. $-\frac{\pi}{k} < Arg(z) \leq \frac{\pi}{k}$.
(c) Let $z = re^{i\phi}$. Then $Log(z) = ln(r) + i\phi$ and $exp(Log(z)) = e^{ln(r)+i\phi}$, so $exp(Log(z)) = z$ for all z.
(d) Let $z = x + iy$. Then $exp(z) = e^x e^{iy}$ and $Log(exp(z)) = ln(e^x) + iArg(e^{iy})$, so $Log(exp(z)) = z$ if and only if $-\pi < y \leq \pi$.
(e) $Log(z)$ is differentiable and a primitive of z^{-1} exactly on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, so $Log'(z) = \frac{1}{z}$ if and only if $Arg(z) \neq \pi$.

2. (a) Let $z_1, \cdots, z_r$ be distinct points and $m_i \in \mathbb{Z}$. Let $D \subset \mathbb{C} \setminus \{z_1, \cdots, z_r\}$ be a domain. Consider a rational function $f : D \to \mathbb{C}$

$$f(z) = (z - z_1)^{m_1} \cdots (z - z_r)^{m_r}.$$

Prove that there exists a branch of $log(f)$ if and only if

$$m_1 I_\gamma(z_1) + \cdots + m_r I_\gamma(z_r) = 0$$

for every closed path γ in D.
(b) Decide whether or not there exists a branch of $log(f_i)$ on D_i for

- $f_1(z) = \frac{z-z_1}{z-z_2}$,

- $f_2(z) = (z - z_1)^{m_1}(z - z_2)^{m_2}(z - z_3)^{m_3}$,
- $f_3(z) = (z - z_1)(z - z_2)$.

Solution. (a) The function f is rational function that is everywhere defined on D, so it is holomorphic. We can use the criterion that there is a branch of $log(f)$ on D if and only if the integral $\int_\gamma \frac{f'}{f} dz = 0$ for every closed path γ in D. We have

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{r} \frac{m_i}{z - z_i}$$

an thus if γ is a closed path in D,

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{r} m_i I_\gamma(z_i).$$

This shows that a branch of $log(f)$ exists if and only if $\sum_{i=1}^{r} m_i I_\gamma(z_i)$.
(b-1) Because z_1, z_2 are in the same connected component of $\mathbb{C} \setminus D_1$, they are also.in the same connected component of $\mathbb{C} \setminus \gamma$ for every closed path γ in D_1. Because $I_\gamma(z)$ is constant on connected components of $\mathbb{C} \setminus \gamma$, we have $I_\gamma(z_1) = I_\gamma(z_2)$, hence

$$m_1 I_\gamma(z_1) + m_2 I_\gamma(z_2) = I_\gamma(z_1) - I_\gamma(z_2) = 0.$$

By (a), there exists a branch of $log(f_1)$.
(b-2) We claim that $I_\gamma(z_2) = I_\gamma(z_3)$ and $I_\gamma(z_1) = 0$ for every closed path γ in D_2. The first part follows from the same argument as before. Since z_1 lies in the unbounded component of $\mathbb{C} \setminus \gamma$, $I_\gamma(z_1) = 0$. There exists a branch of $log(f_2)$ if and only if $m_2 + m_3 = 0$.
(b-3) Let γ be any path that revolves (say, positively) around z_1 exactly once and that does not revolve around z_2. Then $m_1 I_\gamma(z_1) + m_2 I_\gamma(z_2) = 1 \neq 0$, so there does not exist a branch of f_3 on D_3.

3. Prove that $f(z) = \sqrt{\frac{z-1}{z+1}}$ has a branch in $D = \mathbb{C} \setminus [-1, 1]$.

Solution. There exists a branch of $Log(\frac{z-1}{z+1})$ in the domain. Indeed $[-1, 1]$ belongs to the same connected component of $\mathbb{C} \setminus \gamma$ for any γ in D. Thus $I_\gamma(1) = I_\gamma(-1)$. Let g be such a branch, then $e^{\frac{1}{2}g(z)}$ is a branch of $\sqrt{\frac{z-1}{z+1}}$.

4. (a) An open set $D \subset \mathbb{C}$ is *star-shaped* if there exists a point $z_0 \in D$ such that for any $z \in D$, the straight line segment between z and z_0 is contained in D. Prove that a star-shaped open set is simply connected.
(b) Give an example of open set that are not star-shaped but simply connected.

Solution.(a) We may assume $z_0 = 0 \in \mathbb{C}$. For any close path γ in D,let $H(t,s) = s\gamma(t)$ $0 \leq s \leq 1$. Then H is a homotopy between γ and the constant path at 0. Therefore D is simply connected.
(b) For instance, take $D = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.