Complex Analysis Exercise 9 (Solution)

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1. Let $\gamma : [a, b] \to \mathbb{C}$ and $\lambda : [b, c] \to \mathbb{C}$ be two C^1 curves such that $\gamma(b) = \lambda(b)$. Prove that there exists a reparametrization of the curve $\gamma + \lambda$ which is C^1 (Here we allow a reparametrization such that $\gamma'(t_0) = 0$)
Solution. Take a nondecreasing smooth function $phi : [a, c] \to [a, c]$ such that $\phi(a) = a, \phi(c) = c$ and $\phi'(b) = b$. Then $(\gamma + \lambda) \circ \phi$ is C^1.

2. (a) Prove that the sequence $f_n(z) = z^n$, $n \geq 1$ converges locally uniformly but not uniformly on $\{z : |z| < 1\}$.
(b) Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary (not necessarily continuous) function and for $n \in \mathbb{N}$ define $f_n : \mathbb{C} \to \mathbb{C}$

$$f_n(z) = \begin{cases} f(z) & \text{if } |z| \leq n, \\ 0, & \text{if } |z| > n. \end{cases}$$

Show that the sequence (f_n) converges pointwise and locally uniformly to f, and that it converges uniformly to f, if and only if $\lim_{|z|\to\infty} f(z) = 0$.
(c) Give an example of a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions $\mathbb{C} \to \mathbb{C}$ that converges to some function $f : \mathbb{C} \to \mathbb{C}$ that is not continuous.
(d) Give an example of a sequence $(f_n)_{n\in\mathbb{N}}$ of differentiable functions $f_n : \mathbb{R} \to \mathbb{R}$ that converges to a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} f(x)dx.$$

Solution. (a) Because $z^n \to 0$, $n \to \infty$ for every $|z| < 1$, $f_n \to 0$ pointwise. Convergence is not uniform since $sup_{|z|<1}|f_n(z) - 0| = 1$. Recall that locally uniform convergence is equivalent to uniform convergence on compact subsets. Let K be a compact subset of the open unit disk. Put $r = max_{z\in K}|z|$. Then

$$max_{z\in K}|f_n(z) - 0| = max_{z\in K}|z^n| = r^n \to 0, \ \ n \to \infty.$$

Therefore $f_n \to 0$ uniformly on K and f_n converges locally uniformly.
(b) Let $K \subset \mathbb{C}$ be a compact subset and let $r = max_{z\in K}|z|$. Then for all $z \in K$, the sequence $f_n(z)$ becomes stationary and equal to $f(z)$ for $n \geq r$. So (f_n)

converges uniformly on compact subset of $\mathbb{C}$ and it converges locally uniformly on $\mathbb{C}$.

Moreover, (f_n) converges uniformly on $\mathbb{C}$ if and only if

$$lim_{n\to\infty}sup_{|z|>n}|f(z)| = 0,$$

Because $|f_n(z) - f(z)| = 0$, if $|z| \leq n$ and $|f_n(z) - f(z)| = |f(z)|$, if $|z| > n$, this is equivalent to

$$lim_{n\to\infty}sup_{|z|>n}|f(z)| = 0,$$

which is just saying $lim_{n\to\infty}f(z) = 0$.
(c) Let $f_n : \mathbb{C} \to \mathbb{R} \subset \mathbb{C}$ such that $f_n(z) = min\{|z^n|, 1\}$. All f_n are continuous but the limit

$$lim_{n\to\infty}f_n(z) = f(z) = \begin{cases} 0, & \text{if } |z| < 1, \\ 1, & \text{if } |z| \geq 1, \end{cases}$$

is not continuous.
(d) Consider a bump function

$$f_n(x) = \begin{cases} 0, & \text{if } x < n, \\ 30(x-n)^n - 60(x-n)^3 + 30(x-n)^4, & \text{if } n \leq x \leq n+1 \\ 0, & \text{if } x > n+1. \end{cases}$$

Then (f_n) converges pointwise to the constant zero function, but $\int_{-\infty}^{\infty} f_n(x)dx = 1$ for all n.

3. Prove that the series $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ converges locally uniformly on the domain $\{z : Re(z) > 1\}$. What is the derivative $\zeta'(z)$ in terms of a series expansion?

Solution. Let K be a compact subset in $\{z : Re(z) > 1\}$. Denote $a = min_{z\in K}Re(z) > 1$. Then

$$|\frac{1}{n^z}| \leq |\frac{1}{n^a}|$$

in K. $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges for every $a > 1$ by the integral criterion. Therefore $\zeta(z)$ converges uniformly on K by the Weierstrass criterion and it converges locally uniformly in the domain.

From this follows that ζ is holomorphic. One checks that $\frac{d}{dz}\frac{1}{n^z} = -ln(n)\frac{1}{n^z}$ and that the series

$$\xi(z) := \sum_{n=1}^{\infty} -ln(n)\frac{1}{n^z}$$

converges locally uniformly in the domain. Therefore ξ defines a holomorphic function. Morefore, since the partical sum $f_k = \sum_{n=1}^{k} \frac{1}{n^z}$ is a primitive of $f_k' = \sum_{n=1}^{k} -\frac{ln(n)}{n^z}$ and because locally uniform convergence preserves path integrals, $\zeta(z)$ is a primitive of $\xi(z)$.

4. Let f be a holomorphic function on $D = \{z : |z| < 1\}$ with $f(0) = 0$. Prove that the series $\phi(z) = \sum_{n=1}^{\infty} f(z^n)$ converges locally uniformly on D.
Solution. We first prove that ϕ converges uniformly on $\{|z| \leq R\}$. For $|z| < \frac{1}{2}$,

$$|f(z)| = |\int_{[0,z]} f'(t)dt + f(0)| \leq M|z|$$

where $M = max_{|t|<\frac{1}{2}}|f'(t)|$. Take N be an integer bigger than $log_R(\frac{1}{2})$. Then

$$\phi(z) = \sum_{n=1}^{N} f(z^n) + \sum_{n=N+1}^{\infty} f(z^n)$$

which converges uniformly on $|z| \leq R$ by the Weierstrass criterion. Let $K \subset D$ be a compact subset. Then there exists $R < 1$ such that $K \subset \{|z| \leq R\}$. Therefore ϕ converges unifomly on each compact subset K and it converges locally uniformly on D.

5. (a) Prove that the power series $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} : \mathbb{C} \to \mathbb{C}$ converges absolutely and locally uniformly on $\mathbb{C}$.
(b) Is the converse of Weierstrass M-test true in general?
Solution. (a) We can use the Weierstrass M-test. Let $K \subset \mathbb{C}$ be a compact subset and let $r = max\{|z| : z \in K\}$. Then

$$|f(z)| \leq \sum_{n=0}^{\infty} \frac{r^n}{n!} = e^r.$$

(b) No, the converse of Weierstrass M-test may not be true. Take a summation of constant functions $\sum_{n=1}^{\infty} a_n$ where $a_{2n-1} = n$ and $a_{2n} = (-1)n$.