Algebraic Topology **Exercise Sheet 11**

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Question 1:

Compute $H_*(S^n - X)$ where X is a subspace of S^n homeomorphic to $S^k \sqcup S^t$.

Solution:

Recall from the lecture that for any embedding $S^k \to S^n$ we have

$$\tilde{H}_i(S^n - S^k) \cong \tilde{H}_i(S^{n-k-1}).$$

We will compute $\tilde{H}_i(S^n - X)$. Recall that for a topological space Z we have $H_i(Z) \cong \tilde{H}_i(Z)$ for $i \neq 0$ and $H_0(Z) \cong \tilde{H}_0(Z) \oplus \mathbb{Z}$.

Let A be the image of S^k and B be the image of S^t. Observe that $k < n$ and $t < n$.

Firstly, if $n = 1$, then $S^1 - X$ is homotopic equivalent to a discrete set of 4 points. Thus $H_i(S^1 - X) \cong H_i(\sqcup_{j=1}^4 \{*_j\})$. So, assume $n \geq 2$.

Consider the spaces $S^n - A$ and $S^n - B$ A Mayer-Vietoris sequence yields:

$$\cdots \to \tilde{H}_{i+1}(S^n) \to \tilde{H}_i(S^n - X) \to \tilde{H}_i(S^n - A) \oplus \tilde{H}_i(S^n - B) \to \tilde{H}_i(S^n) \to \cdots$$

For $0 \leq i < n-1$ we obtain $\tilde{H}_i(S^n - X) \cong \tilde{H}_i(S^n - A) \oplus \tilde{H}_i(S^n - B)$. Hence

$$\tilde{H}_i(S^n - X) = \begin{cases} \mathbb{Z}^2 & \text{if } k = t \text{ and } i = n-k-1 \\ \mathbb{Z} & \text{if } k \neq t, \quad i = n-k-1 \text{ or } i = n-t-1 \\ 0 & \text{otherwise} \end{cases}$$

For $i \geq n$ we have $\tilde{H}_i(S^n - X) \cong 0$.

Thus, we need to separately analyse the cases $i = n-1$. We have

$$0 \to \tilde{H}_n(S^n) \to \tilde{H}_{n-1}(S^n - X) \to \tilde{H}_{n-1}(S^n - A) \oplus \tilde{H}_{n-1}(S^n - B) \to \tilde{H}_{n-1}(S^n)$$

and hence

$$0 \to \mathbb{Z} \to \tilde{H}_{n-1}(S^n - X) \to \tilde{H}_{n-1}(S^n - A) \oplus \tilde{H}_{n-1}(S^n - B) \to 0$$

If $k = 0$, we have that $n-k-1 = n-1$. Thus: $\tilde{H}_{n-1}(S^n - X) \cong \mathbb{Z}^{\alpha+1}$, where α counts how many between A and B are homeomorphic to S^0.

Question 2:

Compute $H_*(S^n - X)$ where X is a subspace of S^n homeomorphic to $S^k \vee S^t$.

Solution:
We will follow the same idea of the previous question. Again, we can assume $n > 1$, since for $n = 1$ we have that $S^1 - X$ is homotopic equivalent to three points.

Let A be the image of S^k and B the image of S^l. Let $\{p\} = A \cap B$. Let $Y = S^n - \{p\}$. Observe that $Y \cong \mathbb{R}^n$. It is an easy exercise in point-set topology that $Y - ((A \cup B) - \{p\})$ is homeomorphic to $S^n - X$, and similarly $Y - (A - \{p\}) \cong S^n - A$ and $Y - (B - \{p\}) \cong S^n - B$. To simplify notation, we will write $Y - A$ to mean $Y - (A - \{p\})$, and similarly for B. Using a Mayer-Vietoris sequence on Y with open sets $Y - A$ and $Y - B$ we have:

We have:

$$\cdots \to \tilde{H}_{i+1}(Y) \to \tilde{H}_i(Y - X) \to \tilde{H}_i(Y - A) \oplus \tilde{H}_i(Y - B) \to \tilde{H}_i(Y) \to \cdots$$

and so

$$\cdots \to 0 \to \tilde{H}_i(Y - X) \to \tilde{H}_i(Y - A) \oplus \tilde{H}_i(Y - B) \to 0$$

Thus

$$\tilde{H}_i(S^n - X) \cong \tilde{H}_i(S^{n-k-1}) \oplus \tilde{H}_i(S^{n-l-1}).$$

Question 3:

Let X be the 1-skeleton of the 4-dimensional simplex (also known as the complete graph on 5 vertices). Show that X cannot be embedded in S^2. *Note: Be careful to make your proof fully rigorous!*

Solution:
Denote $1, 2, 3, 4, 5$ the five vertices of X. Suppose there was an embedding $f\colon X \to S^2$. By Jordan's Theorem, the S^1 corresponding to the cycle $(1, 2, 3)$ separates S^2 into two path-components A, B. Without loss of generality, we can assume that $f(4) \in A$. Since f is continuous, A is path connected and the interior of a 1-cell is connected, we obtain that the interior of the 1-cells $(1, 4), (2, 4), (3, 4)$ is contained in A.

Consider the S^1 corresponding to the cycle $(1,2,4)$. Again, it separates S^2 in two path-components C, D. Up to swapping the roles of C, D, we can assume that $C \subseteq A$. Indeed, suppose that both component intersected both A and B. Let γ_C and γ_D be paths inside C, D respectively that join a point of A and a point of B in $S^2 - f((1,2,4))$. Let b_C, b_D be the endpoints. Since B is path connected, there must be a path $\eta \subseteq B$ between b_C, b_D. This means that η must intersect $f((1,2,4)$. Since $f(1,2)$ is not contained in B, η must intersect $f((1,2)) \cup f(2,4)$. Since η is contained in B, we obtain that one between $f((1,2))$ or $f((2,4))$ intersects B, which is a contradiction.

So, $C \subseteq A$. An analogous argument shows that there are two regions E, F obtained from the embeddings $f((1,3,4))$ and $f((2,3,4))$ that are contained in A.

The fact that $f(4) \in A$ also implies that $f(5) \in A$. Otherwise, the image of the 1-cell $(4,5)$ would a path between A, B, disjoint from $f((1,2,3))$. This implies that $f((1,2,3)) \subseteq A \cup B$, contradicting A, B are path components of $A \cup B$.

Now, suppose $f(5) \in C$. By a similar argument as above, there needs to be a path between $f(5)$ and $f(3)$. However, such a path contradicts the fact that X is embedded. Repeating this argument for E, F yields to a contradiction.