Algebraic Topology **Exercise Sheet 12**

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Question 1:

Let K be any triangulation of Δ_n, and let $g\colon K \to \Delta_n$ be a simplicial approximation to the identity. Show that the number of simplices of K that map onto Δ_n is odd. (Hint: g must carry $\partial\Delta_n$ into itself.)

Solution:
Let ∂K be the set of simplices of K that are contained in $\partial\Delta_n$ (K is a triangulation of Δ_n. Consider a simplex $\sigma = \langle v_0, \ldots, v_n\rangle$ of K. We claim that the boundary of σ needs to map to the boundary of Δ_n. Indeed, let $e_0, \ldots, e_n$ be the 0–simplices of Δ_n. Since g is simplicial, $g(v_i) \subseteq \{e_0, \ldots, e_n\}$ for all i. Note that in general $g(v_i)$ does not need to be distinct from $g(v_j)$. This means that the image of a face of σ is a simplex spanned by at most $n-1$ vertices of Δ_n. In particular, it is contained in $\partial\Delta_n$.

A consequence of the above argument is that $g(\partial K) \subseteq \partial\Delta_n$. Observe that, by definition of approximation to the identity, g is homotopic to the identity *relative to the boundary*. This is not hard to see if you look explicitly at how the homotopy is defined in the lecture.

Let q be the map that collapses the boundary of Δ_n and K (remember that K is a triangulation of Δ_n).

$$\begin{array}{ccc} (K, \partial K) & \xrightarrow{\quad g \quad} & (\Delta_n, \partial\Delta_n) \\ \big\downarrow q & & \big\downarrow q \\ (K/\partial K, \{*\}) & \xrightarrow{\ G\ } & (\Delta_n/\partial\Delta_n, \{*\}) \end{array}$$

Since $g(\partial K) \subseteq \partial\Delta_n$, the diagram commutes. When we look at homology, since g is homotopic to the identity relative to the boundary, it induces an isomorphism in homology. The induced map G is a map between spheres, since g is an isomorphism in homology, G must have degree ± 1. For each simplex σ of K, let $G_\sigma\colon K/\partial K \to \Delta_n/\partial\Delta_n$ be the map defined as G on $q(\sigma)$ and constant $\{*\}$ on the rest of $K/\partial K$. This is well defined since g sends the boundary of an n–simplex to $\partial\Delta_n$, which gets then mapped to $\{*\}$ by q.

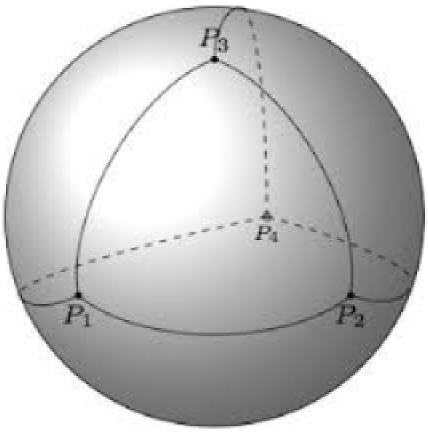
Thus,

$$\deg(G) = \sum_{\sigma \in K} \deg(G_\sigma).$$

To compute $\deg(G_\sigma)$ observe that G_σ is constant to $\{*\}$ if $\sigma = \langle v_0, \ldots, v_n \rangle$ is not mapped to an n-simplex. This is to say, if there are i, j such that $g(v_i) = g(v_j)$. Otherwise, $G_\sigma)$ is an homeomorphism, and hence has degree ± 1. Since $\sum_{\sigma \in K} \deg(G_\sigma) = \pm 1$, we have that there needs to be an odd number of simplices $\sigma = \langle v_0, \ldots, v_n \rangle$ such that $g(v_i) \neq g(v_j)$ for all $i \neq j$, i.e. an odd number of simplices of K that maps onto Δ_n.

Question 2:

Triangulate S^2 via the central projection of a regular tetrahedron T inscribed in S^2. Show that there is no simplicial approximation $T \to T$ (without subdividing) to the antipodal map.



Solution:
Let A be the antipodal map, and suppose that g is an approximation to A. Consider the point $A(P_4)$, which is in the interior of the simplex $\langle P_1, P_2, P_3 \rangle$. Note that $g(A(P_4)) \subseteq \text{carrier}(AA(P_4)) = \{P_4\}$. This implies that $g(P_1) = g(P_2) = g(P_3) = P_4$. In particular, g is not surjective. This means that the image of g is contractible, showing that g is homotopic to a constant map. But this contradicts the fact that g is homotopic to A.