Algebraic Topology **Exercise Sheet 13**
Prof. Dr. Alessandro Sisto
Assistant: Davide Spriano

Let A be a finitely generated abelian group. The *tensor product* $A \otimes \mathbb{Q}$ is the abelian group generated by the symbols $a \otimes k$ for $a \in A$ and $q \in \mathbb{Q}$ subject to the relations

$$\begin{aligned} a \otimes q + b \otimes q &= (a+b) \otimes q \\ a \otimes q + a \otimes k &= a \otimes (q+k) \end{aligned}$$

Note: $\mathbb{Q}$ can be substituted with $\mathbb{R}$ in this exercise sheet.

Question 1:

Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of finitely generated abelian groups. Show that

$$0 \to A \otimes \mathbb{Q} \to B \otimes \mathbb{Q} \to C \otimes \mathbb{Q} \to 0$$

is exact. In particular, conclude that if $C_\bullet$ is a chain complex of finitely generated abelian groups, then $C_\bullet \otimes \mathbb{Q}$ is a chain complex.

Question 2:

Let $C_\bullet$ be a chain complex of finitely generated abelia groups. Show that

$$H_n(C_\bullet \otimes \mathbb{Q}) \cong H_n(C_\bullet) \otimes \mathbb{Q}.$$

Conclude that if X is a CW complex such that $X^{(n)}$ has finitely many cells for every n, then

$$H_n(X; \mathbb{Q}) \cong H_n(X) \otimes \mathbb{Q}.$$