Algebraic Topology **Exercise Sheet 13**

Prof. Dr. Alessandro Sisto
Assistant: Davide Spriano

Let A be a finitely generated abelian group. The *tensor product* $A \otimes \mathbb{Q}$ is the abelian group generated by the symbols $a \otimes k$ for $a \in A$ and $q \in \mathbb{Q}$ subject to the relations

$$a \otimes q + b \otimes q = (a+b) \otimes q$$
$$a \otimes q + a \otimes k = a \otimes (q+k)$$

Note: $\mathbb{Q}$ can be substituted with $\mathbb{R}$ in this exercise sheet.

Question 1:

Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of finitely generated abelian groups. Show that

$$0 \to A \otimes \mathbb{Q} \to B \otimes \mathbb{Q} \to C \otimes \mathbb{Q} \to 0$$

is exact. In particular, conclude that if $C_\bullet$ is a chain complex of finitely generated abelian groups, then $C_\bullet \otimes \mathbb{Q}$ is a chain complex.

Solution:
We start by recalling that for any finitely generated abelian group Z, any $z \in Z$ and $q \in \mathbb{Q}$ we have

$$0 = 0 \otimes q = z \otimes 0$$

which implies

$$-(z \otimes q) = (-z) \otimes q = z \otimes (-q).$$

Indeed, observe that $-(z \otimes 0) = (-z) \otimes 0$. Then

$$\begin{aligned} z \otimes 0 = (2z) \otimes 0 + (-z) \otimes 0 = z \otimes 0 + z \otimes 0 - (z \otimes 0) = \\ = z \otimes (0+0) - (z \otimes 0) = 0. \end{aligned}$$

The argument for $0 \otimes q = 0$ is symmetric.

Let f be the map $A \to B$ and g be the map $B \to C$. Since $g \circ f = 0$, it follows that the composition $(g \otimes \mathrm{Id}) \circ (f \otimes \mathrm{Id})$ is equal to zero. Also, $g \otimes \mathrm{Id}$ is surjective. Indeed, consider $c \otimes q \in C \otimes \mathbb{Q}$. Since g is surjective there is b such that $g(b) = c$. Then $g \otimes \mathrm{Id}(b \otimes q) = c \otimes q$. To show that $f \otimes \mathrm{Id}$ is injective and that $\ker(g \otimes \mathrm{Id}) \subseteq \mathrm{Im}(f \otimes \mathrm{Id})$ we will need to recall more properties of the tensor product.

Let Z be a finitely generated abelian group. Then Z can be written as the direct sum of the free and torsion part, namely $Z \cong \mathbb{Z}^{n_Z} \oplus T_Z$, for some finite group T_Z and integer n_Z. We claim that the tensor product sends the torsion part to zero, i.e. $Z \otimes \mathbb{Q} \cong \mathbb{Z}^{n_Z} \otimes \mathbb{Q}$. Indeed, let $z \in Z$ be a torsion element. Then there exists an integer C such that $Cz = 0$. Then for each $q \in \mathbb{Q}$ we have

$$z \otimes q = C\left(z \otimes \frac{q}{C}\right) = (Cz) \otimes \frac{q}{C} = 0.$$

Let $e_1, \ldots, e_{n_Z}$ be a basis of $\mathbb{Z}^{n_Z}$. Then every element $z \otimes q$ of $\mathbb{Z}^{n_Z} \otimes \mathbb{Q}$ can be written as

$$z \times q = \sum_{i=1}^{n_Z} (z_i e_i \otimes q) = \sum_{i=1}^{n_Z} e_i \otimes (z_i q).$$

In particular, every element is identified by a n_Z-tuple of elements of $\mathbb{Q}$. This provides an identification $\mathbb{Z}^{n_Z} \otimes \mathbb{Q} \cong \mathbb{Q}^{n_Z}$.

Recall that f is the map $A \to B$ and g is the map $B \to C$. Since f is injective, it send non-torsion elements to non-torsion elements. In particular, f restricts to an injective map $f|_{\mathbb{Z}^{n_A}} \colon \mathbb{Z}^{n_A} \to \mathbb{Z}^{n_B}$. It is straightforward to verify that the induced map $\mathbb{Q}^{n_A} \to \mathbb{Q}^{n_B}$ is injective.

We now show exactness in $B \otimes \mathbb{Q}$. Since the original sequence was exact, we obtain $n_A + n_C = n_B$. Moreover, since$f \otimes \mathrm{Id}$ is injective and $\mathrm{Im}(f \otimes \mathrm{Id}) \subseteq \ker(g \otimes \mathrm{Id})$ we have that the dimension of $\ker(g \otimes \mathrm{Id})$ is at least n_A. Since the map $g \otimes \mathrm{Id}$ is surjective, the rank-nullity theorem yields $\dim(\ker(g \otimes \mathrm{Id})) = n_C$, and the result follows.

Question 2:

Let $C_\bullet$ be a chain complex of finitely generated abelian groups. Show that

$$H_n(C_\bullet \otimes \mathbb{Q}) \cong H_n(C_\bullet) \otimes \mathbb{Q}.$$

Conclude that if X is a CW complex such that $X^{(n)}$ has finitely many cells for every n, then

$$H_n(X; \mathbb{Q}) \cong H_n(X) \otimes \mathbb{Q}.$$

Solution:
$C_\bullet$ has the form:

$$\cdots \to C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \to \cdots$$

We claim that there are isomorphisms $\ker(f_n)\otimes\mathbb{Q} \to \ker(f_n \otimes \mathrm{Id})$ and $\mathrm{Im}(f_n)\otimes\mathbb{Q} \to \mathrm{Im}(f_n \otimes \mathrm{Id})$ for all n. Assuming the claim, for all n the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Im}(f_{n+1}) \otimes \mathbb{Q} \arrow[r] \arrow[d, "\cong"] & \ker(f_n) \otimes \mathbb{Q} \arrow[r] \arrow[d, "\cong"] & H_n(C_\bullet) \otimes \mathbb{Q} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d, "\cong"] \\
\mathrm{Im}(f_{n+1} \otimes \mathrm{Id}) \arrow[r] & \ker(f_n \otimes \mathrm{Id}) \arrow[r] & H_n(C_\bullet; \mathbb{Q}) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Since rows are exact, we get the result by the 5 lemma.

To prove the claim, consider a map between finitely generated abelian groups $f\colon A \to B$ (in our case it will be $f_n\colon C_n \to C_{n-1}$). There is a short exact sequence:

$$0 \to \ker(f) \to A \to A/\ker(f) \cong \mathrm{Im}(f) \to 0.$$

This implies

$$\mathrm{rk}(\ker(f)) + \mathrm{rk}(\mathrm{Im}(f)) = \mathrm{rk}(A).$$

Similarly, we will have

$$\mathrm{rk}(\ker(f \otimes \mathrm{Id})) + \mathrm{rk}(\mathrm{Im}(f \otimes \mathrm{Id})) = \mathrm{rk}(A \otimes \mathbb{Q}).$$

In the previous Question we proved that $\mathrm{rk}(A) = \mathrm{rk}(A \otimes \mathbb{Q})$. We claim $\mathrm{rk}(\ker(f \otimes \mathrm{Id})) \geq \mathrm{rk}(\ker(f))$. Indeed, write $\ker(f) = \mathbb{Z}^k \oplus T_k$, and let $a_1, \dots a_k$ be a basis element of $\mathbb{Z}^k$. Observe that $a_1 \otimes 1, \dots, a_k \otimes 1$ are elements of $\ker(f \otimes \mathrm{Id})$, since $f(a_i) \otimes 1 = 0 \otimes 1 = 0$. We claim that they are linearly independent vectors. If they were not, the matrix with integer coefficient $[a_1 \cdots a_k]$ would have a maximal minor with determinant zero. Thus, $a_1, \dots, a_k$ could not be generators of $\mathbb{Z}^k$, which concludes the claim. An analogous argument shows $\mathrm{rk}(\mathrm{Im}(f \otimes \mathrm{Id})) \geq \mathrm{rk}(\mathrm{Im}(f))$. Together with the previous equalities we obtain $\mathrm{rk}(\ker(f \otimes \mathrm{Id})) = \mathrm{rk}(\ker(f))$ and $\mathrm{rk}(\mathrm{Im}(f \otimes \mathrm{Id})) = \mathrm{rk}(\mathrm{Im}(f))$. It is now straightforward to verify that the maps in the above commutative diagram are isomorphism.