**Algebraic Topology** **Exercise Sheet 1**

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# Question 1:

Let $X$ be a topological space, and suppose $X = \sqcup_{i\in I} X_i$, where each $X_i$ is a path-connected component. Show that $H_n(X) = \bigoplus_{i\in I} H_n(X_i)$.

**Solution:**
Recall that for a topological space $Y$ we denote with $\Delta_n(Y)$ the free abelian group with basis the singular $n$–simplices of $Y$. Recall that an element of $\Delta_n(Y)$ is an $n$–chain, i.e. a finite formal sum $\rho = \sum a_\sigma \sigma$, where $a_\sigma \in \mathbb{Z}$ and $\sigma$ is a singular $n$–simplex, i.e. a map $\sigma\colon \Delta_n \to Y$. Observe that since $X_i \subseteq X$, every singular simplex of $X_i$ is also a singular simplex of $X$. Moreover, given a finite collection of singular chains $\{\rho_i\}$ such that $\rho_i \in \Delta_n(X_i)$, we can associate to them an element $\rho$ of $\Delta_n(X)$ simply as $\rho = \sum \rho_i$. Since each $\rho_i$ is a finite sum, so it is $\rho$. Observe that the group $\bigoplus_{i\in I} \Delta_n(X_i)$ consists of finite sums $\sum \sigma\rho_i$, where $\rho_i \in \Delta_n(X_i)$. By the above reasoning we can define a group homomorphism

$$f\colon \bigoplus_{i\in I} \Delta_n(X_i) \to \Delta_n(X).$$

From the definition of $\Delta_n(X)$, it follows that $f$ is injective. For a chain to be equal to zero, it is necessary that the sum of the coefficients of each singular $n$–simplex is zero. But since for $i \neq j$ the spaces $X_i$ and $X_j$ are disjoint, no cancellations are possible between the elements of $\Delta_n(X_i)$ and $\Delta_n(X_j)$.

Note that for this part we did not use any assumption on path connectivity. Indeed, whenever there is a topological space $Y$ that can be written as $Y = \sqcup Y_i$, there is always an injective homomorphism $f\colon \bigoplus_{i\in I} \Delta_n(Y_i) \to \Delta_n(Y)$. *Warning! This does not imply that there is an injective map between the homologies! This is a good check to see if you understood the definition of homology.*

The connectivity assumptions will allow us to show that in this case it is an isomorphims, namely

$$f\colon \bigoplus_{i\in I} \Delta_n(X_i) \xrightarrow{\cong} \Delta_n(X).$$

Recall that if $Y$ is a path connected topological space and $h\colon Y \to Z$

is a continuous map, then the image of $Y$ is contained in a path-connected component of $Z$.

Consider an $n$–chain $\rho = \sum n_\sigma \sigma$. Recall that $\sigma$ is a singular $n$–simplex, i.e. a map $\sigma\colon \Delta_n \to X$. Since $\Delta_n$ is path-connected, the image of $\sigma$ is contained in some $X_i$. For every $i \in I$, let $\rho_i$ be the $n$–chain obtained considering only the singular simplices of $\rho$ with image in $X_i$. We have

$$\rho = \sum \rho_i.$$

Since $\rho$ is a finite sum, there are only finitely many $\rho_i$, each of which is a finite sum. This shows that every $\rho \in \Delta_n(X)$ is in the image of $f$, showing that $f$ is surjective, thus an isomorphism.

We will now show that this induces an isomorphism between homologies. Since $\partial$ is a group homomorphism, we have

$$\partial \left( \sum a_\sigma \sigma \right) = \sum a_\sigma \left( \partial \sigma \right)$$

Thus $\partial \rho = \sum \partial \rho_i$. We claim that $\partial \rho = 0$ if and only if $\partial \rho_i = 0$ for all $i$. Consider two singular simplices $\sigma_1$ and $\sigma_2$ and suppose that they share a face. This means that $\sigma_1^{(i)} = \sigma_2^{(j)}$ as functions. Thus, since their images intersect, they belong to the same connected component. In particular, there cannot be cancellations between elements of $\partial \rho_i$ and $\partial \rho_j$ for $i \neq j$, which proves the claim.

Similarly, if the image of $\sigma$ is contained in $X_i$, so it is the image of $\partial \sigma$. Thus, a singular chain $\rho$ is a cycle if and only if all $\rho_i$ are, and similarly for boundaries. In particular, this shows that cycles and boundaries of $X$ are exactly the sums of cycles and boundaries of $X_i$, showing that two cycles of $X$ differ by a boundary (i.e. they represent the same homology class) if and only if each of their $X_i$ components differ by a boundary. This concludes the proof.