Algebraic Topology **Exercise Sheet 2**
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Question 1:

Let X be a topological space and $f\colon [0,1] \to X$ be a path (recall, this means f is continuous). Show that $f + f^{-1}$ is a 1–boundary in two different ways:

1. combining lemmas proven in class,
2. finding an explicit 2-chain whose boundary is $f + f^{-1}$.

Solution:
Solution 1: Observe that $f+f^{-1}$ is a boundary if and only if $[\![f]\!]+[\![f^{-1}]\!] = 0$ in $H_1(X)$. By Lemma 2 of the lecture we have $[\![f]\!] + [\![f^{-1}]\!] = [\![f * f^{-1}]\!]$. Let $x = f(0)$. Then $f * f^{-1}$ is homotopic relative to x to the constant path c_x. By Lemma 1 we have $[\![f * f^{-1}]\!] = [\![c_x]\!]$. By Lemma 0, $[\![c_x]\!] = 0$, which concludes the first proof.

Solution 2: Let $\sigma\colon \Delta_2 \to X$ be the singular simplex defined as follows:

$$\sigma(\lambda_0 e_0 + \lambda_1 e_1 + \lambda_2 e_2) = f(\lambda_1),$$

and let $\sigma_0\colon \Delta_2 \to X$ be the simplex with constant value $f(0)$. Let $\rho = \sigma + \sigma_0$. We claim that $\partial\rho = f + f^{-1}$, which provides the result. Firstly, observe that $\partial\sigma = f + f^{-1} - c_{f(0)}$. Indeed, let s_0, s_1 be a basis for Δ_1. Then

$$\begin{aligned}
\sigma^{(0)}&\colon ((1-t)s_0 + ts_1) \mapsto (0e_0 + (1-t)e_1 + te_2) \mapsto f(1-t)\\
\sigma^{(1)}&\colon ((1-t)s_0 + ts_1) \mapsto ((1-t)e_0 + 0e_1 + te_2) \mapsto f(0)\\
\sigma^{(2)}&\colon ((1-t)s_0 + ts_1) \mapsto ((1-t)e_0 + te_1 + 0e_2) \mapsto f(t).
\end{aligned}$$

Reasoning as in Question 2, we obtain that $\partial\sigma_0 = c_{f(0)}$. Thus, $\partial\tau = f^{-1} + f - c_{f(0)} + c_{f(0)} = f^{-1} + f$. *Attention!* It is not true that $\partial\sigma = f + f^{-1}$, because it is not true that $f^{-1} + f + c_{f(0)} = f^{-1} + f$ in $\Delta_1(X)$. The equality only holds in $H_1(X)$.

Question 2:

Let X be a topological space, $x \in X$ a point and for each n consider the constant singular simplex $c_n \colon \Delta_n \to X$ with constant value x (that is the map that sends every point of Δ_n to x).

When is c_n a boundary?

Solution:

First, we compute when c_n is a cycle. Since c_n is the constant simplex on x, all of its faces are c_{n-1}, i.e. the constant function $\Delta_{n-1} \to \{x\}$. Then for $n \geq 1$ we have

$$\partial c_n = \sum_{i=0}^{n} (-1)^i c_n^{(i)} = \sum_{i=0}^{n} (-1)^i c_{n-1} = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ c_{n-1} & \text{if } n \text{ is even.} \end{cases}$$

Thus, for n odd we have that $c_n = \partial c_{n+1}$ showing that it is a boundary. For n even, we have that c_n is not even a cycle, in particular not a boundary. Thus c_n is a boundary precisely when it is a cycle and precisely in odd degrees.

Question 3:

Let F_2 be the free group on two generators. Show that the abelianization of F_2 is $\mathbb{Z}^2$. As a remark, observe that the proof also works to show that the abelianization of F_n is $\mathbb{Z}^n$.

Solution:

Let F_2 be generated by a, b and let $\mathbb{Z}^2$ be generated by x, y. Recall that F_2 is defined as follows: the elements of F_2 are words in the letters a, b, a^{-1}, b^{-1}, under the equivalence relation $uaa^{-1}v \sim ua^{-1}av \sim ubb^{-1}v \sim ub^{-1}bv \sim uv$, where u and v are any words.

Given an element $g \in F_2$, let $\phi_a(g)$ be the sum with sign of all the occurrences of the letter a in a word w representing g. Note that the equivalence relation preserves the exponent sum, so if $w' \sim w$ both represent the element g, then their exponent sum will be the same. Let ϕ_b be defined similarly for b. Consider the map

$$\begin{aligned} \phi \colon F_2 &\to \mathbb{Z}^2 \\ g &\mapsto (\phi_a(g), \phi_b(g)). \end{aligned}$$

We claim that it is a group homomorphism. Let $g, h \in F_2$, choose a representative v for g and w for h, and consider vw. It is clear that the number of occurrences of the letter a in vw precisely coincide with the number of occurrences of a in v plus the ones in w, and similarly for a^{-1}. Since vw is a representative of gh, and since the exponent sum does not depend on the choice of representative, we get

$$\phi_a(gh) = \phi_a(g) + \phi_a(h),$$

and similarly for ϕ_b. Thus ϕ is a group homomorphism. We claim that the kernel of ϕ coincides with the commutator subgroup $[F_2, F_2]$. Indeed, this would imply $F_2/[F_2, F_2] \cong \mathbb{Z}^2$.

The kernel of ϕ consists of those elements of F_2 which are both in the kernel of ϕ_a and ϕ_b, namely those elements g that can be represented by a word which has the same number of occurrences of a, a^{-1} and the same number of occurrences of b, b^{-1}, for example $aab^{-1}a^{-1}a^{-1}b$. Let's call this set $\sum$. Observe that $[F_2, F_2] \leq \sum$. Indeed, $[F_2, F_2]$ is generated by elements of the form $ghg^{-1}h^{-1}$, which is not hard to see that belong to $\sum$. Now, let $g \in F_2$ be such that $g \in \sum$ and let w be a shortest representative of g. We will show by induction on the length of w that it can be written as a product of commutators. First observe that w needs to have even length.

Base case: If the length of w is less or equal to 4, and w is minimal in its equivalence class, then w is either empty of a commutator (this can be done by hand by just listing all words of length at most 4 with zero exponent sum).

Induction step: The strategy is going to be the following. We will find z which represents an element of $[F_2, F_2]$ such that, after performing reduction, wz is a shorter word with exponent sum zero. The induction hypothesis then yields $wz \in [F_2, F_2]$. Since we picked $z \in [F_2, F_2]$, we conclude that $w \in [F_2, F_2]$, which will complete the proof.

To simplify notation, let $A = a^{-1}$ and $B = b^{-1}$. Consider w with zero exponent sum. If all the letters of w are in $\{a, A\}$, the fact that w has zero exponent sum and it is the shortest representative yield that w is the empty word, and similarly for $\{b, B\}$. Then, up to exchanging the roles of a and A, we can write w as $tauAv$, for some words t, u and v. Let $z = [v^{-1}a, u^{-1}]$. Then: We have

$$wz = tauAv[v^{-1}a, u^{-1}] = tauAvv^{-1}au^{-1}Avu = tvu$$

It is straightforward to verify that tvu has still exponent sum zero, concluding the proof.