Algebraic Topology **Exercise Sheet 3**

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Let X, Y be topological spaces and $f\colon X \to Y$ be a continuous map. We will use the notation $f_\#\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ to denote the induced map on the level of fundamental groups and $f_*\colon H_n(X) \to H_n(Y)$ the induced map on the level of homology.

Question 1:

[Problem 2, Page 177, Bredon] If $f\colon X \to Y$ is a map and $f(x_0) = y_0$ show that the diagram

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_\#"] \arrow[d, "\phi_X"] & \pi_1(Y, y_0) \arrow[d, "\phi_Y"] \\ H_1(X) \arrow[r, "f_*"] & H_1(Y) \end{tikzcd}$$

commutes, where ϕ_X and ϕ_Y are the Hurewicz homomorphisms.

Solution:
Let $s_0 \in S^1$ be a basepoint and let $\gamma\colon S^1 \to X$ be such that $\gamma(s_0) = x_0$. We want to show that $\phi_Y \circ f_\#([\gamma]) = f_* \circ \phi_X([\gamma])$.

Firstly, recall how the Hurewicz map is defined. Let $\Delta_1 = [0, 1]$ and let $q\colon \Delta_1 \to S^1 \cong [0, 1]/0 \sim 1$ be the quotient map. Let (Z, z_0) be a pointed topological space and $\tau\colon S^1 \to Z$ be a representative of a class of $\pi_1(Z, z_0)$. Then $\phi_Z([\tau])$ is defined to be the homology class of the singular simplex $\Delta_1 \xrightarrow{q} S^1 \xrightarrow{\tau} Z$. I.e. $\phi_Z([\tau]) = [\![\tau \circ q]\!]$.

Then we have

$$\phi_Y \circ f_\#([\gamma]) = \phi_Y([f \circ \gamma]) = [\![f \circ \gamma \circ q]\!].$$

Similarly

$$f_* \circ \phi_X([\gamma]) = f_*([\![\gamma \circ q]\!]) = [\![f \circ \gamma \circ q]\!].$$

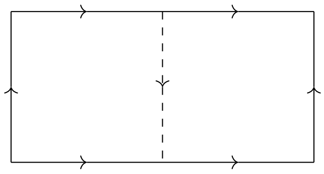
Question 2:

[Problem 3, Page 177, Bredon] If $f\colon X \to Y$ is a covering map then $f_\#$ is an injective homomorphism by covering space theory. Is it true that $f_*\colon H_1(X) \to H_1(Y)$ is also injective? Give either a proof or a counterexample.

Solution:
The statement is not true, and we will give two counterexamples.

Conuterexample 1: Let X be any path connected topological space such that $H_1(X) \neq \{1\}$, for example $X = S_1$. Let $Y = X_1 \sqcup X_2$, where X_i are copies of X and let $f\colon Y \to X$ be defined to be the identity $X_i \to X$ on each copy of X inside Y. It is straightforward to verify that f is a covering map. Then we have that $H_1(Y) = H_1(X_1) \bigoplus H_1(X_2)$ and, since f restricts to the identity on each copy of X inside Y, it follows that $f_*|_{H_1(X_i)}\colon H_1(X_i) \to H_1(X)$ is the identity. Thus, the map $f_*\colon H_1(X_1) \bigoplus H_1(X_2) \to H_1(X)$ is not injective.

Counterexample 2: We will now provide an example where Y is connected. The torus is a 2-fold cover of the Klein Bottle, as illustrated by the following diagram.



Providing a more detailed proof of this fact is beyond the scope of this exercise sheet. If you want to see more details, some can be found at the following link `https://math.stackexchange.com/questions/1073425/two-sheeted-covering-of-the-klein-bottle-by-the-torus`.

Let T be the torus and K the Klein bottle. Then $H_1(T) = \mathbb{Z} \bigoplus \mathbb{Z}$ and $H_1(K) = \mathbb{Z} \bigoplus \mathbb{Z}/2\mathbb{Z}$. There can be no injective map between $H_1(T)$ and $H_1(K)$. Indeed, let s, t be generators of the two $\mathbb{Z}$ factors of $H_1(T)$, let a be a generator of the $\mathbb{Z}$ factor of $H_1(K)$ and b a generator of the $\mathbb{Z}/2\mathbb{Z}$ factor.

If an element of H_1 is of infinite order, then it must be of the form $ka + \varepsilon b$, for $k \in \mathbb{Z} - \{0\}$ and $\varepsilon \in \{1, 0\}$. Thus, if $h \in H_1$ is of infinite order, then $h + h$ has the form $2ka$, because $\varepsilon b + \varepsilon b = 2\varepsilon b = 0$. Suppose there was an injective map $j\colon H_1(T) \to H_1(K)$. Then $j(s)$ and $j(t)$ are of infinite order, and there are $n, m \neq 0$ such that $j(2s) = na$ and $j(2t) = ma$. Then $2ms - 2nt$ is a non trivial element of $H_1(T)$ whose image is trivial in $H_1(K)$.

Question 3:

[Problem 1, Page 182, Bredon] Multiplication by a non-zero integer $n\colon \mathbb{Z} \to \mathbb{Z}$ fits in a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

Use this to derive the exact sequence

$$0 \to \frac{H_n(X)}{pH_n(X)} \to H_n(X;\mathbb{Z}/p\mathbb{Z}) \to \ker\{p\colon H_{n-1}(X) \to H_{n-1}(X)\} \to 0.$$

Solution:
We will prove the exercise assuming the following theorem:

Theorem 3.1. *Let $A_\bullet$, $B_\bullet$ and $C_\bullet$ be chain complexes, and suppose that there is a short exact sequence*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0.$$

Then there is a long exact sequence

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to \cdots$$

Recall that $H_n(A)$ is defined to be the n-th homology of the chain complex $A_\bullet$. That is, $A_\bullet$ has the form

$$\cdots A_{n+1} \xrightarrow{\partial^A_{n+2}} A_{n+1} \xrightarrow{\partial^A_{n+1}} A_n \xrightarrow{\partial^A_n} A_{n-1} \xrightarrow{\partial^A_{n-1}} \cdots$$

and $H_n(A)$ is the group $\ker(\partial^A_n)/\mathrm{Im}(\partial^A_{n+1})$.

Assuming the theorem, let $q\colon \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be the quotient map. Our first step is to show that there is a short exact sequence of chain complexes

$$0 \to \Delta(X) \xrightarrow{p} \Delta(X) \xrightarrow{q} \Delta(X) \otimes \mathbb{Z}/p\mathbb{Z} \to 0.$$

This amounts to show that for every n the squares of the following diagram commutes, and that the rows are short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \Delta_n(X) \arrow[r, "p"] \arrow[d, "\partial"] & \Delta_n(X) \arrow[r, "q"] \arrow[d, "\partial"] & \Delta_n(X) \otimes \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & \Delta_{n-1}(X) \arrow[r, "p"] & \Delta_{n-1}(X) \arrow[r, "q"] & \Delta_{n-1}(X) \otimes \mathbb{Z}/p\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We will adopt the convention to denote with a upper bar the class of a number in $\mathbb{Z}/p\mathbb{Z}$, for instance $\bar{3}$ represents the equivalence class of 3 modulo p, and we will also denote with p both the map and the number $p \in \mathbb{Z}$. Then the maps p and q are defined on the generators as follows. Let σ be a singular n–simplex. Then $p(\sigma) = p \cdot \sigma$ and $q(\sigma) = \bar{1}\sigma$. To check that the squares commutes it suffices to just write down explicitly all the maps. To see that the sequences are exact, first observe that p is injective and q is surjective. Now let $\rho = \sum a_\sigma \sigma$ be a chain such that $q(\rho) = \sum \bar{a_\sigma}\sigma = 0$ in $\Delta_n(X) \otimes \mathbb{Z}/p\mathbb{Z}$. Then, by definition of $\Delta_n(X) \otimes \mathbb{Z}/p\mathbb{Z}$ we need to have that $\bar{a_\sigma} = \bar{0}$ for all a_σ. This amounts to say that all a_σ are divisible by p, thus ρ was in the image of $p\colon \Delta_n(X) \to \Delta_n(X)$. Hence, the sequence is short exact. Since n was generic, we obtain the claim.

Using the theorem, there exist a long exact sequence

$$\cdots H_{n+1}(X;\mathbb{Z}/p\mathbb{Z}) \to H_n(X) \xrightarrow{p} H_n(X) \xrightarrow{q} H_n(X;\mathbb{Z}/p\mathbb{Z}) \to H_{n-1}(X) \to \cdots$$

First, this implies that $\frac{H_n(X)}{pH_n(X)} \to H_n(X;\mathbb{Z}/p\mathbb{Z})$ is injective. Indeed, the long exact sequence yields

$$\ker\left(H_n(X) \to H_n(X;\mathbb{Z}/p\mathbb{Z})\right) = pH_n(X). \tag{1}$$

By exactness again we obtain

$$\mathrm{Im}(H_n(X;\mathbb{Z}/p\mathbb{Z}) \to H_{n-1}(X)) = \ker(H_{n-1}(X) \xrightarrow{p} H_{n-1}(X)), \tag{2}$$

yielding that $H_n(X;\mathbb{Z}/p\mathbb{Z}) \to \ker\{p\colon H_{n-1}(X) \to H_{n-1}(X)\}$ is surjective.

Using equation (2) again, we have that an element $a \in H_n(X;\mathbb{Z}/p\mathbb{Z})$ gets mapped to zero in $H_{n-1}(X)$ if and only if it gets mapped to zero in $\ker(H_{n-1}(X) \to H_{n-1}(X))$. Similarly, equation (1) gives that an element $a \in H_n(X;\mathbb{Z}/p\mathbb{Z})$ is in the image of $H_n(X)$ if and only if it is in the image of $\frac{H_n(X)}{pH_n(X)}$. Thus, exactness in the long exact sequence yields exactness in the desired short exact one.