Algebraic Topology **Exercise Sheet 4**

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The final goal of this exercise sheet is to compute $H_2(T)$, where $T = S^1 \times S^1$ is the torus. You don't need to solve the exercises in the proposed order, you can skip one and use it in the following ones.

In what follows, H_* is a homology theory. *Note: Unless specified you cannot use results about singular homology, but only results that follows from the axioms of homology theory.*

Question 1:

Let X be a topological space, $Y \subseteq X$ a subspace and suppose that there is a retraction $r\colon X \to Y$, i.e. a continuous map such that $r|_Y = \mathrm{Id}_Y$. Let $i\colon Y \to X$ be the inclusion. Show that $i_*\colon H_n(Y) \to H_n(X)$ is injective.

Question 2:

Let $T = S^1 \times S^1$ be the torus. Write $S^1 = [-1, 1]/(-1 \sim 1)$ and consider the subspace $B = S^1 \times [0, 1] \subseteq T$. Let $j\colon B \to T$ be the inclusion.

1. Let H_* be any homology theory. Use the previous exercise to show that $j_*\colon H_*(B) \to H_*(T)$ is injective.
2. (Bonus) Let H_* be the singular homology. Use the Hurewicz map to show that $j_*\colon H_1(B) \to H_1(T)$ is injective.

Question 3:

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{x^2 + y^2 + z^2} = 1\}$ and let $U = S^2 \cap \{(x, y, z) \in \mathbb{R}^3 \mid |z| \geq \frac{1}{2}\}$. Intuitively, U consists of two discs, one around the north pole and the other around the south pole. Compute $H_2(S^2, U)$.

Question 4:

Use excision twice to show $H_*(T, B) \cong H_*(S^2, U)$, and then compute $H_2(T)$.