Algebraic Topology **Exercise Sheet 4**

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The final goal of this exercise sheet is to compute $H_2(T)$, where $T = S^1 \times S^1$ is the torus. You don't need to solve the exercises in the proposed order, you can skip one and use it in the following ones.

In what follows, H_* is a homology theory. *Note: Unless specified you cannot use results about singular homology, but only results that follows from the axioms of homology theory.*

Question 1:

Let X be a topological space, $Y \subseteq X$ a subspace and suppose that there is a retraction $r\colon X \to Y$, i.e. a continuous map such that $r|_Y = \mathrm{Id}_Y$. Let $i\colon Y \to X$ be the inclusion. Show that $i_*\colon H_n(Y) \to H_n(X)$ is injective.

Solution:
Since r is a retraction, we have $r \circ i\colon Y \to Y = \mathrm{Id}_Y$. Thus, for each n we have that the composition

$$H_n(Y) \xrightarrow{i_*} H_n(X) \xrightarrow{r_*} H_n(Y)$$

is the identity on $H_n(Y)$. Suppose that i_* is not injective. Then there is $0 \neq a \in H_n(Y)$ such that $i_*(a) = 0$. Thus $r_* \circ i_*(a) = 0$, contradicting that $r_* \circ i_*$ is the identity.

Question 2:

Let $T = S^1 \times S^1$ be the torus. Write $S^1 = [-1, 1]/(-1 \sim 1)$ and consider the subspace $B = S^1 \times [0, 1] \subseteq T$. Let $j\colon B \to T$ be the inclusion.

1. Let H_* be any homology theory. Use the previous exercise to show that $j_*\colon H_*(B) \to H_*(T)$ is injective.

2. (Bonus) Let H_* be the singular homology. Use the Hurewicz map to show that $j_*\colon H_1(B) \to H_1(T)$ is injective.

Solution:
Part 1: Consider the function $r\colon T \to B$ defined as $r(x,y) = (x,|y|)$. Since $r|_B = \mathrm{Id}_B$, we have that r is a retraction. The results then follows from Question 1.

Part 2: Recall that the following diagram commutes:

$$\begin{tikzcd} \pi_1(B) \arrow[r, "j_\#"] \arrow[d, "\phi_B"] & \pi_1(T) \arrow[d, "\phi_T"] \\ H_1(B) \arrow[r, "j_*"] & H_1(T) \end{tikzcd}$$

The space B deformation retracts on $S^1 \times \{0\}$, thus $\pi_1(B) \cong \pi_1(S^1) \cong \mathbb{Z}$. Moreover, $\pi_1(T) \cong \pi_1(S^1 \times S^1) \cong \pi_1(S^1 \times \{0\}) \oplus \pi_1(\{0\} \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. Since both groups are abelian, the Hurewicz maps are isomorphisms. We only need to show that $j_\#$ is injective. Let $\alpha_1, \alpha_2\colon S^1 \to T$ be defined as $\alpha_1(x) = (x,0)$, and $\alpha_2(x) = (0,x)$. Then α_1 and α_2 are generators of $\pi_1(T)$. Let $\gamma\colon S^1 \to B$ be defined as $\gamma(x) = (x,0)$. Then $[\gamma]$ generates $\pi_1(B)$ and $j \circ \gamma = \alpha_1$. Thus, for all $m \neq 0$ we have $j_\#([\gamma]^m) = ([\alpha_1]^m) \neq 0$, which shows that $j_\#$ is injective.

Question 3:

Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{x^2+y^2+z^2} = 1\}$ and let $U = S^2 \cap \{(x,y,z) \in \mathbb{R}^3 \mid |z| \geq \frac{1}{2}\}$. Intuitively, U consists of two discs, one around the north pole and the other around the south pole. Compute $H_2(S^2, U)$.

Solution:
Observe that $U \cong \mathbb{D}^2 \sqcup \mathbb{D}^2$, and recall that we know that $H_2(\mathbb{D}^2) \cong H_1(\mathbb{D}^2) \cong 0$, and $H_2(S^2) \cong \mathbb{Z}$. The long exact sequence of the pair yields

$$\cdots \to H_2(U) \to H_2(S^2) \to H_2(S^2, U) \to H_1(U) \to \cdots$$

and so

$$\cdots \to 0 \to \mathbb{Z} \to H_2(S^2, U) \to 0 \to \cdots$$

Thus $H_2(S^2, U) \cong H_2(S^2) \cong \mathbb{Z}$.

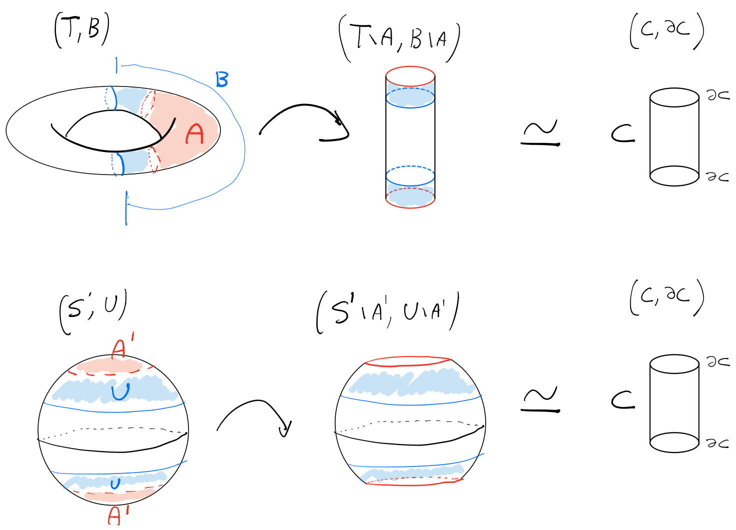
Question 4:

Use excision twice to show $H_*(T, B) \cong H_*(S^2, U)$, and then compute $H_2(T)$.

Solution:
Let $C = S^1 \times [0, 1]$ be the cylinder, and let $\partial C = S^1 \times \{0, 1\}$ be its boundary. Let $A = S^1 \times \left(\frac{1}{3}, \frac{2}{3}\right) \subseteq B \subseteq T$. We have that the closure of A is contained in the interior of B. Using excision, we have $H_*(T - A, B - A) \cong H_*(T, B)$. Observe that $(T - A, B - A)$ is homotopic equivalent to $(C, \partial C)$.

Now, consider $A' = S^2 \cap \{(x, y, z) \in \mathbb{R}^3 \mid |z| > \frac{2}{3}\} \subseteq U \subseteq S^2$, where U and S^2 are defined as in Question 3. As above, we have that the closure of A' is contained in the interior of U. Thus $H_*(S^2 - A', U - A') \cong H_*(S^2, U)$. Again, observe that $(S^2 - A', U - A')$ is homotopic equivalent to $(C, \partial C)$. Chaining all the above together we obtain:

$$H_*(T, B) \cong H_*(T - A, B - A) \cong H_*(C, \partial C) \cong H_*(S^2 - A', U - A') \cong H_*(S^2, U)$$



We will now compute $H_2(T)$. The long exact sequence of the pair (T, B) yields:

$$\cdots \to H_2(B) \to H_2(T) \to H_2(T, B) \to H_1(B) \to H_1(T) \to \cdots.$$

By Question 2 the map $H_1(B) \to H_1(T)$ is injective and hence we have $H_2(T) \cong H_2(T, B)$. By Question 3 we have $H_2(S^2, U) \cong \mathbb{Z}$, and hence $H_2(T) \cong H_2(T, B) \cong H_2(S^2, U) \cong \mathbb{Z}$.