Algebraic Topology **Exercise Sheet 5**

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Note: with the notation $X \sqcup Y$, we mean the topological space where the sets X, Y are clopen, and the inclusions $X \to X \sqcup Y$, $Y \to X \sqcup Y$ are homeomorphisms on their images.

Question 1:

Let X_i be topological spaces and $X = \bigsqcup X_i$. Let $A_i \subseteq X_i$ be a family of subsets and for each i let $j_i \colon X_i \to X$ be the inclusion.

1. For H_* the singular homology, show that $\bigoplus (j_i)_* \colon \bigoplus H_p(X_i, A_i) \to H_p(X, \bigcup A_i)$ is an isomorphism.

2. Prove the following lemma and use it to show that (1) holds for any homology theory.

Lemma 1.1 (5-lemma)**.** *Consider the following commutative diagram of abelian groups:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads, "r"] & B \arrow[r] \arrow[d, "s", "\cong"'] & C \arrow[r] \arrow[d, "t"] & D \arrow[r] \arrow[d, "u", "\cong"'] & E \arrow[d, hook, "v"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

If the rows are exact, the maps s, u are isomorphisms, r is surjective and v is injective, then t is an isomorphism.

Solution:
Part 1: We will show that for each n the inclusion j_i induces an isomorphism $\oplus(j_i)_* \colon \bigoplus(\Delta_n(X_i)/\Delta_n(A_i)) \to \Delta_n(X)/\Delta_n(\bigcup A_i)$. Since the set of the $(j_i)_*$s forms a chain map, this implies that the homologies of the two chain complexes are isomorphic via the maps $(j_i)_*$. By the definition of $\Delta_n(Z)$, the only thing that we need to show is that the inclusion maps induce isomorhpisms $\bigoplus \Delta_n(X_i) \cong \Delta_n(X)$ and $\bigoplus \Delta_n(A_i) \cong \Delta_n(A)$. This was done in Exercise Sheet 1, Question 1, yielding the result.

5-lemma: With an abuse of notation, we will call all horizontal maps ∂. Let $0 \neq c \in C$ be a group element. We will show that $t(c) \neq 0$. Let $\partial(c) \in D$. If $\partial(c) \neq 0$, so $u(\partial(c)) \neq 0$. However, by commutativity of the diagrams, we have$\partial(t(c)) \neq 0$, and so $t(c) \neq 0$. So suppose $\partial(c) = 0$. By exactness, there is $0 \neq b \in B$ such that $\partial(b) = c$. Consider $s(b)$. If $\partial(s(b)) \neq 0$, then we are done by commutativity. Otherwise, there is a' such that $\partial(a') = s(b)$. Since r is surjective and s is an isomorphism, there exists $a \in A$ such that $\partial(r(a)) = s(b)$, and so $\partial(a) = b$. However, this implies $c = \partial\partial a = 0$, which is a contradiction.

The argument to show that t is surjective is analogous.

Part 2: Let $A = \bigcup A_i$. By the additivity axiom, the inclusions induce isomorphisms$\bigoplus H(X_i) \to H(X)$ and $\bigoplus H(A_i) \to H(\bigcup A$. The long exact sequence of the pair yields:

$$\begin{array}{ccccccccc}
\bigoplus H(A_i) & \longrightarrow & \bigoplus H(X_i) & \longrightarrow & \bigoplus H(X_i, A_i) & \longrightarrow & \bigoplus H(A_i) & \longrightarrow & \bigoplus H(X_i) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \oplus(j_i)_* & & \downarrow \cong & & \downarrow \cong \\
H(A) & \longrightarrow & H(X) & \longrightarrow & H(X, A) & \longrightarrow & H(A) & \longrightarrow & H(X)
\end{array}$$

The result now follows from 5-lemma.

Question 2:

Consider a CW-complex K, possibly not of finite dimension. For the singular homology, show that the inclusion i of $K^{(n)}$ in K induces isomorphisms $i_* \colon H_p(K^{(n)}) \to H_p(K)$ for all $p < n$.

Hints:

1. You can use the following fact: for $m \geq n+1$ the following is an isomorphism:

$$H_n(K^{(n+1)}) \xrightarrow{i_*} H_n(K^{(m)}).$$

2. Use fact (1) to show that the following is an isomorphism:

$$H_n(K^{(n+1)}) \xrightarrow{i_*} H_n(K).$$

Solution:
We to indicate the inclusion from the n to m skeleton we use $i_{n,m}$, with the convention $K^{(\infty)} = K$. We start by showing that $(i_{n+1,\infty})*\colon H_n(K^{(n+1)}) \to H_n(K)$. is surjective. Let $c = \sum n_\sigma \sigma$ be a singular chain of K. We know that there exists N such that every singular simplex σ of c has image in $K^{(N)}$. Let $[\![c]\!]_\infty$ denote the class of c in $H_n(K)$ and $[\![c]\!]_N$ the class of c in $H_n(K^{(N)})$. Observe that this is well defined. If c is a homology class of K, it means that $\partial c = 0$, which is still true in the chain complex of $K^{(N)}$, since the image of ∂c is contained in the image of c. Let $i_{n+1,N}$ be the inclusion $K^{(n+1)} \to K^{(N)}$. By fact one, it induces an isomorphism at the level of homology. Thus, there is c' such that $(i_{n+1,N})([\![c']\!]_{n+1}) = [\![c]\!]_N$. Note that the composition $i_{N,\infty} \circ i_{n+1,N}$ coincides with the inclusion $i_{n+1,\infty}\colon K^{(n+1)} \to K$. In particular

$$(i_{n+1,\infty})_*([\![c']\!]_{n+1}) = [\![c]\!]_\infty.$$

To show injectivity, let c' be singular chain of $K^{(n+1)}$ such that $\partial c' = 0$, and assume that $i_*([\![c']\!]_{n+1}) = 0$ in $H_n(K)$. Then there exists a singular chain c of K such that $\partial c = (i_{n+1,\infty})_* c'$. Reasoning as before, we need to have that c is contained in $K^{(N)}$, for some N. We use this fact to show that $\partial c = c'$, yielding the result.

To conclude the exercise, consider the following commutative diagram, where 3 out of 4 maps are isomorphisms.

$$\begin{tikzcd}
H_p(K^{(p+1)}) \arrow[r, "i_{p+1,n}"] \arrow[d, "i_{p+1,\infty}"] & H_p(K^{(n)}) \arrow[d, "i_{n,\infty}"] \\
H_p(K) \arrow[r, "\mathrm{Id}"] & H_p(K)
\end{tikzcd}$$

Question 3:

Let A, B, C be abelian groups, and consider the following short exact sequence:

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0.$$

1. Suppose that there is a splitting $p\colon C \to B$, i.e. a map such that $j \circ p\colon C \to C = \mathrm{Id}_C$. Show that $B \cong A \oplus C$.

2. Suppose that there is a splitting $p\colon B \to A$, i.e. a map such that $p \circ i\colon A \to A = \mathrm{Id}_A$. Show that $B \cong A \oplus C$.

Solution:

Let $r = p \circ j \colon B \to B$. Let $b \in B$ and write

$$b = (b - r(b)) + r(b).$$

Observe that $r(b) \in p(C)$ and $b - r(b) \in \ker(r)$. This gives us an injective map $f \colon B \to p(C) \bigoplus \ker(r)$. Indeed, suppose there are b, b' such that $r(b) = r(b')$ and $b - r(b) = b' - r(b')$. Then $b = b'$.

By the property of p, we have that p is injective, and thus $\mathrm{Im}(p) \cong C$ and $\ker(r) = \ker(j)$. By exactness i is injective and thus $\ker(r) = \ker(j) = \mathrm{Im}(i) \cong A$. This gives us an injective map $f \colon B \to A \bigoplus C$. We need to show that the map is surjective. Let $a \in a$ and $c \in C$ and consider $b = i(a) + p(c)$. Then $r(i(a) + p(c)) = p(c)$ and leading that $f(i(a) + p(c)) = a + c$.

The other case is completely analogous, just consider $s = j \circ p$ instead of r, and exchange the roles of A and C.

Alternatively, after constructing in a similar way a map from B to $A \bigoplus C$, one can consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A \arrow[r] & A \bigoplus C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

and observing that the two rows are exact, all vertical maps but the central are isomorphisms, and the diagram commutes. Then the result follows from the 5-lemma. (Note, this is probably more work than how it is written above).