Algebraic Topology **Exercise Sheet 8**

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Question 1:

[Hatcher Ex 1 p. 155] Construct a surjective map $S^n \to S^n$ of degree zero, for each $n \geq 1$.

Solution:
For notational purposes, we denote by $\bar{x}$ a vector of $\mathbb{R}^{n+1}$ and $\bar{y}$ a vector of $\mathbb{R}^n$. Thus we will write $\bar{x} = (\bar{y}, x_{n+1})$.

Consider $S^n = \{\bar{x} \in \mathbb{R}^{n+1} \mid \|\bar{x}\| = 1\}$, the upper emisphere $U = \{\bar{x} = (x_1, \ldots, x_{n+1}) \mid x_{n+1} \geq 0\}$, and the lower emisphere $L = \{\bar{x} = (x_1, \ldots, x_{n+1}) \mid x_{n+1} \leq 0\}$.Let $i\colon U \to S^n$ be the inclusion, and let $f\colon S^n \to S^n$ be the map that "slides" the upper emisphere down the sphere and collapse the lower emisphere to a point. In formulas:

$$f(\bar{y}, x_{n+1}) = \begin{cases} \left(\bar{y}\sqrt{1 - \left(2\left(x_{n+1} - \frac{1}{2}\right)\right)^2}, 2\left(x_{n+1} - \frac{1}{2}\right)\right) & \text{if } x_{n+1} \geq 0 \\ (0, \ldots, 0, -1) & \text{if } x_{n+1} \leq 0 \end{cases}$$

Note that f restricted to the upper emisphere is surjective. Let $R\colon S^n \to S^n$ be the inversion of the last coordinate, namely $R(\bar{y}, x_{n+1}) = (\bar{y}, -x_{n+1})$. Let $\Phi\colon S^n \to S^n$ be defined as

$$\Phi(\bar{x}) = \begin{cases} f(\bar{x}) & \text{if } \bar{x} \in U \\ f \circ R(\bar{x}) & \text{if } \bar{x} \in L \end{cases}$$

Since f is surjective, so it is Φ. We claim that the degree of Φ is zero. Let Φ_1 be defined as Φ on $\mathring{U}$ and constant on the complement, and Φ_2 as Φ on $\mathring{L}$ and constant on the complement. Since $\Phi_2 = \Phi_1 \circ R$, we obtain that Φ_1 and Φ_2 have opposite degrees, yielding that Φ has degree 0.

Question 2:

[Hatcher Ex 9 p. 156] Compute the homology of the following 2-complexes:

1. The quotient of S^2 obtained identifying north and south poles to a point.
2. The space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of $D2$ and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.

Solution:
See the separate file for this solution.

Question 3:

[Hatcher Ex 12 p. 156] Show that the quotient map $S^1 \times S^1 \to S^2$ obtained by collapsing the subspace $S^1 \vee S^1$ to a point is not null homotopic (i.e. homotopic to a constant map) by showing that it induces an isomorphism on H_2.

Bonus: On the other hand, show via covering spaces that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.

Solution:
Consider the standard CW structure on the torus T, and let σ be the only 2-cell. Observe that $S^1 \vee S^1 = T^{(1)}$. Note that quotient map of the exercise is nothing but the map $q\colon T \to S^2_\sigma$. We showed that the map q induces an isomorphism $q_* H(T, T^{(1)}) \to S^2_\sigma$. Observe that $H_2(T^{(1)}) = 0$, since $T^{(1)}$ does not have cells of dimension higher than 1. Thus the long exact sequence of the pair yields that the inclusion $T \to (T, T^{(1)})$ is injective. Hence, we have

$$\mathbb{Z} \cong H_2(T) \hookrightarrow H_2(T, T^{(1)}) \xrightarrow{\cong} H_2(S^2_\sigma) \cong \mathbb{Z},$$

which shows that the map q_* has infinite image, thus it is not the zero map.

To see that q_* is surjective, observe that $H_1(T, T^{(1)}) = 0$. This can be see using excision to conclude that $H_1(T, T^{(1)}) \cong H_1(D^2, \partial D^2) \cong 0$. We will skip the precise details of this part, which we saw different times: use homotopy equivalence to "thiken" $T^{(1)} = S^{(1)} \vee S^{(1)}$, excide a smaller thickening of $T^{(1)}$ and observe it is homotopic equivalent to $(D^2, \partial D^2)$. Since the only surjective homomorphisms $Z^2 \to Z^2$ are isomorphisms, we get that the map $T^{(1)} \to T$ induces an isomorphism on H_1, yielding the result.

For the bonus part, consider a map $f\colon S^2 \to T$. The map f lifts to a map between the universal covers. The universal cover of S^2 is S^2, and the universal cover of T is $\mathbb{R}^2$. Thus, we can write f as a map $S^2 \to \mathbb{R}^2 \to T$. Since $\mathbb{R}^2$ is contractible, such a map needs to be null-homotopic.