Algebraic Topology **Exercise Sheet 9**

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Question 1:

Consider the cycle a given by the inclusion $\partial\Delta^2 \to \Delta^2$, and let b be its barycentric subdivision. Show explicitly that a and b differ by a relative boundary, i.e. write an element $c \in \Delta_2(\Delta^2)$ such that $\partial c - (a-b) \in \Delta_2(\partial\Delta^2)$. You can present a solution by pictures.

Solution:
Due to a typo, the exercise was much easier than planned. The intended statement is that a is the identity $\Delta^2 \to \Delta^2$ (and $c \in \Delta_3(\Delta^2)$). As it is written, we already have $(a-b) \in \Delta_1(\partial\Delta^2)$. However, the exercise is still interesting if we want c such that $\partial c - (a-b) = 0$, and not just in $\Delta_1(\partial\Delta^2)$.

The solution to the intended exercise and of the simpler version are attached in the other file.

Question 2:

Let X be a topological space and Y be a subspace of X. Let $i\colon Y \to X$ be the inclusion. Let C be space obtained from X by "attaching a cone on Y". More precisely, is the topological space

$$C = (Y \times [0,1]) \sqcup X/\sim$$

where $\sim$ is defined as $(y,0) \sim i(y)$ and $(y,1) \sim (y',1)$ for all $y, y' \in Y$. Compute $H_*(C)$.

Solution:
To solve this exercise, we will use excision. Let Z be the image of $Y \times [0,1]$ in C and let p be the point corresponding to $Y \times \{1\}$. It is standard point set topology to show that (C, Z) is homotopic equivalent to $(C, \{p\})$. Similarly, $(C - \{p\}, Z - \{p\})$ is homotopic equivalent to (X, Y). Using excision on a

neighbourhood of p contained in the interior of Z, we obtain that $(C, \{p\})$ has the same homology of (X, Y). Thus $\tilde{H}_*(C) = H_*(X, Y)$.

Concluding the exercise here is ok. To compute $H_*(C)$, we only need to compute $H_0(C)$. This is much harder to do explicitly. We have a long exact sequence:

$$\cdots \to H_0(X) \to H_0(C) \to H_0(C, X) \to 0.$$

Using excision and homotopy equivalence again, we obtain that $H_0(C, X) \cong H_0(Z, Y)$, which will allow, once we have explicit values, to compute $H_0(C)$.

Question 3:

Show that homotopy equivalence of chain complexes is an equivalence relation.

Solution:

Recall that $A_\bullet$ is chain homotopic to $B_\bullet$ if there are chain maps $f\colon A_\bullet \to B_\bullet$ and $g\colon B_\bullet \to A_\bullet$ and a collection of maps $h_n\colon A_n \to A_{n+1}$, $k_n\colon B_n \to B_{n+1}$ such that

$$\mathrm{Id}_B - f \circ g = \partial k + k\partial$$

and

$$\mathrm{Id}_A - g \circ f = \partial h + h\partial.$$

It is clear that being chain homotopic is a symmetric relation. To see that it is reflexive, take $f = g = \mathrm{Id}_A$ and $h = k = 0$.

We will now show that it is transitive.

Let f, g, h, k be the chain maps and maps associated to a chain homotopy equivalence between $A_\bullet$ and $B_\bullet$ and f', g', h', k' be the ones associated to a chain homotopy equivalence between $B_\bullet$ and $C_\bullet$.

Let

- $F\colon A_\bullet \to C_\bullet$ be defined as $f' \circ f$,
- $G\colon C_\bullet \to A_\bullet$ be defined as $g \circ g'$,
- and $H\colon A_\bullet \to A_{\bullet+1}$ be defined as $h + g \circ h' \circ f$.

We claim that

$$\mathrm{Id}_A - G \circ F = \partial H + H\partial.$$

Recall that g, f, g', f' are chain maps, hence it hold $\partial g = g\partial$. This is not true for h, k, h', k'. Let $a \in A_\bullet$. Then we have:

$$\begin{aligned}
a - G \circ F(a) &= a - g \circ g' \circ f' \circ f(a) = \\
&= a - g\left[g' \circ f'(f(a))\right] = \\
&= a - g\left[f(a) - \partial h'(f(a)) - h'\partial(f(a))\right] = \\
&= a - g \circ f(a) + \partial g \circ h' \circ f(a) + g \circ h' \circ f\partial(a) = \\
&= a - (a - \partial h(a) - h\partial(a)) + \partial g \circ h' \circ f(a) + g \circ h' \circ f\partial(a) = \\
&= \partial(h + g \circ h' \circ f)(a) + (h + g \circ h' \circ f)(\partial a) = \\
&= \partial H(a) + H\partial(a)
\end{aligned}$$

The argument for the other direction is completely analogous.