Algebraic Topology **Mock Exam**

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Question 1:

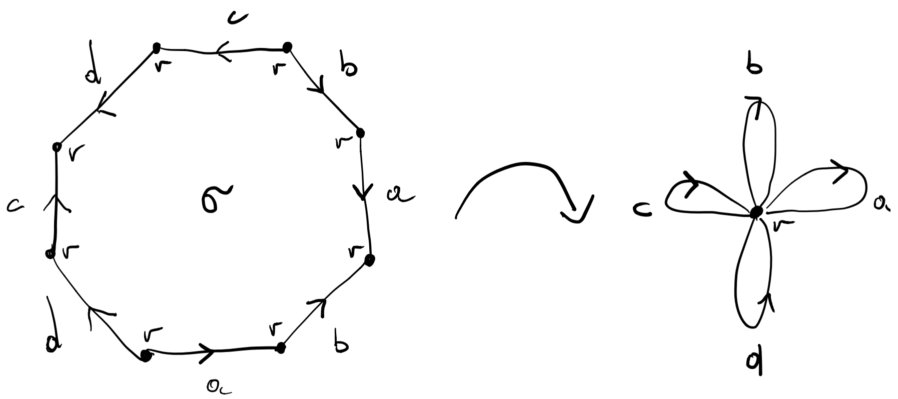
1. State the Hurewicz Theorem about the connection between homology and fundamental group.
2. The proof of Hurewicz Theorem uses a certain map $\Psi\colon \Delta_1(X) \to \pi_1^{\mathrm{ab}}(X)$. Define the map.
3. Show that $\Psi(B_1(X)) = 0$.

Question 2:

1. State the excision Theorem for singular homology.
2. Define the map $\Upsilon\colon L_p(\Delta^q) \to L_p(\Delta^q)$, where $L_p(\Delta^q)$ denotes the affine singular simplices of Δ^q.
3. Proove that Υ is a chain map.

Question 3:

Let Σ_2 be the CW-complex with one 0-cell labelled v, four 1-cells labelled a, b, c, d and one 2-cell labelled σ with attaching map as in the following picture.



1. Compute the singular homology groups $H_*(\Sigma_2)$ of Σ_2.

Solution:
We will compute the cellular homology, which is isomorphic to the singular homology. By the number of cells, we know that the homology is the one of the following chain complex:

$$0 \to \mathbb{Z} \xrightarrow{\beta_2} \mathbb{Z}^4 \xrightarrow{\beta_1} \mathbb{Z} \to 0.$$

Since we have only one 0-cell, the map β_1 is the zero map. We need to compute the map β_2. It is a map from $\mathbb{Z}$ to $\mathbb{Z}^4$ that sends $1 \mapsto ([\sigma : a], [\sigma : b], [\sigma : c], [\sigma : d])$. Recall that $[\sigma : a]$ is the degree of the map $p_a \circ f_\sigma$, where $f_\sigma \colon \partial D^2_\sigma \to \Sigma_2^{(1)}$ is the attaching map of the 2-cell σ, and p_a is the quotient map $\Sigma_2^{(1)} \to S^1_a$. Thinking of $\partial\sigma$ as an octagon as in the picture, we have that $p_a \circ f_\sigma$ sends the six edges labelled b, c or d to a point. Let F_1 be the map that is defined as $p_a \circ f_\sigma$ on the first edge labelled a, and is constant on the other seven edges. Let F_2 be the map that is defined as $p_a \circ f_\sigma$ on the second edge labelled a, and it is constant on the other seven edges. Then $\deg(p_a \circ f_\sigma) = \deg(F_1) + \deg(F_2)$. Observe that $F_2 = A \circ F_1$, where A is the antipodal map. Since $\deg(A) = -1$, we have $\deg(F_1) = -\deg(F_2)$, yielding that $[\sigma : a] = \deg(p_a \circ f_\sigma) = 0$. The exact same argument holds for $[\sigma : b], [\sigma : c]$ and $[\sigma : d]$, and thus β_2 is the zero map.

Then we have $H_0(\Sigma_2) = H_2(\Sigma_2) = \mathbb{Z}$, $H_1(\Sigma_1) = \mathbb{Z}^4$ and $H_n(\Sigma_2) = 0$ for $n \geq 3$.

2. Let X be a finite CW-complex such that $X^{(2)}$ is homeomorphic to Σ_2. What can you say about $H_1(X)$?

Solution:
Recall the following theorem: *Let K be a CW-complex, and let $j \colon K^{(n-1)} \to K^{(n)}$ be the inclusion. Then $j_* \colon H_p(K^{(n-1)}) \to H_p(K^{(n)})$ is*

- *surjective if $p \neq n$;*
- *injective if $p \neq n-1$;*
- *an isomorphism if $p \neq n, n-1$.*

Assume that X has dimension $N \geq 3$. Then the inclusions $X^{(n)} \hookrightarrow X^{(n+1)}$ induce the following sequence of maps:

$$H_1(X^{(2)}) \to H_1(X^{(3)}) \to \cdots \to H_1(X^{(N)}).$$

By the above theorem, all the maps are isomorphism, yielding $H_1(X) \cong H_1(\Sigma_2)$.

3. What can you say about $H_2(X)$?

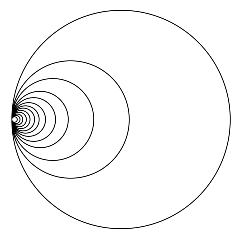
Solution:
We will consider the sequence of inclusions as before, but on the second homology this time:

$$H_2(X^{(2)}) \to H_2(X^{(3)}) \to \cdots \to H_2(X^{(N)}).$$

By the above theorem, the left-most map is surjective, and all the other maps are isomorphisms. In particular, we deduce that there needs to be a surjection $H_2(\Sigma_2) \to H_2(X)$, and hence that $H_2(X)$ is a quotient of $H_2(\Sigma_2)$, so it is isomorphic to either $0, \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for some positive integer n.

Question 4:

For each integer $n \geq 1$, let C_n be the circle of radius $\frac{1}{n}$ centered in $\left(\frac{1}{n}, 0\right)$. For clarity, $C_n = \left\{(x,y) \mid \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2}\right\}$. Let $X = \bigcup_{n=1}^{\infty} C_n \subseteq \mathbb{R}^2$ be endowed with the subspace topology.



1. Show that the singular homology group $H_1(X)$ is not finitely generated. *Possible hint: using suitable maps to the various circles may help*

Solution:
For each $n > 0$, let $C_{\leq n} = \bigcup_{i=1}^{n} C_i$ and let $\phi_n \colon C_{\leq n} \to X$ be the inclusion of the first n circles into X. Observe that $C_{\leq n}$ is a retract of X. Indeed, let $r_n \colon X \to C_{\leq n}$ be the map defined as the identity on C_i, for $i \leq n$ and to be constant $(0,0)$ on C_i, for $i \geq n+1$. Then r_n is a retraction, i.e. $r_n \circ \phi_n = \mathrm{Id}_{C_{\leq n}}$. This implies that the map ϕ_n is injective in homology. Indeed, we have that $(r_n)_* \circ (\phi_n)_*$ is an isomorphism in homology, and thus $(\phi_n)_*$ needs to be injective. Suppose that $H_1(X)$ was finitely generated. Then we could write $H_1(X) = \mathrm{Z}^N \oplus T$, T is a finite abelian group and N is called the rank of $H_1(X)$. We would have:

$$(\phi_{N+1})_* \colon H_1(C_{\leq N+1}) \hookrightarrow H_1(X)$$

We claim that $H_1(C_{\leq N+1}) \cong \bigoplus_{i=1}^{N+1} \mathbb{Z}$. To prove this fact, one can use excision and follow the exact same proof of Ex Sheet 6, Question 2. We provide a quicker proof using induction and Mayer-Vietoris Theorem. We want to show that $H_1(C_{\leq m}) \cong \bigoplus_{i=1}^{m} H_1(C_i)$ for all m. If $m = 1$, the claim is true. Suppose the claim was true for m, we will prove it for $m+1$. Let U, V be contractible open neighborhoods of $(0,0)$ contained in $C_{\leq m}$ and C_{m+1} respectively, for instance $U = \{x \in C_{\leq m} \mid |x| \leq \frac{1}{m}\}$, and $V = \{x \in C_{m+1} \mid |x| \leq \frac{1}{m+1}\}$. Let $A = C_{\leq m} \cup U$ and $B = C_{m+1} \cup V$. Then $A \cup B = C_{\leq m+1}$, A and B are homotopic equivalent to $C_{\leq m}$ and C_{m+1} respectively and $A \cap B$ is contractible. The reduced Mayer-Vietoris sequence yields:

$$\cdots \tilde{H}_1(A \cap B) \to \tilde{H}_1(A) \oplus \tilde{H}_1(B) \to \tilde{H}_1(C_{m+1}) \to \tilde{H}_0(A \cap B) \to \cdots$$

and hence $\tilde{H}_1(C_{\leq m}) \oplus \tilde{H}_1(C_{m+1}) \cong \tilde{H}_1(C_{\leq m+1})$, giving the claim.

Using the claim, we can write the inclusion above as

$$(\phi_{N+1})_* \colon H_1(C_{\leq N+1}) \hookrightarrow H_1(X)$$
$$(\phi_{N+1})_* \colon \bigoplus_{i=1}^{N+1} \mathbb{Z} \hookrightarrow \mathbb{Z}^N \oplus T,$$

which is a contradiction.

2. For each $n > 0$, let $i_n \colon C_n \to X$ be the inclusion. Show that the homorphism

$$\bigoplus (i_n)_* \colon \bigoplus H_1(C_n) \to H_1(X)$$

is not surjective. That is to say that $H_1(X)$ is not generated by the images of the first homology groups of the circles C_n.

Solution:
For each $n > 0$, let $p_n\colon X \to C_n$ be the map that is the identity on C_n and sends all other points to $(0,0)$. Observe that the following diagram commutes:

$$\begin{array}{ccc} C_n & \xrightarrow{i_n} & X \\ & \overset{\mathrm{Id}}{\searrow} & \big\downarrow p_n \\ & & C_n \end{array}$$

Let $\Psi\colon S^1 \to X$ be a map defined as follows. We want to subdivide $[0,1]$ into infinitely many sub intervals I_n, and define $\Psi|_{I_n}$ to be a parametrization of the circle C_n that starts and ends at $(0,0)$. Firstly, let ψ_n be an appropriate parametrization of C_n, for instance the map $[0, 2\pi] \to C_n$ defined as $x \mapsto \frac{1}{n}(1 - \cos(x), \sin(x))$. Then, let $I_n = [1 - 2^{n-1}, 1 - 2^n]$ (we have $I_1 = [0, \frac{1}{2}]$, $I_2 = [\frac{1}{2}, \frac{3}{4}]$ and so on). Define Ψ as

$$\Psi(x) = \psi_n\left(2^{n+1}\pi(x - (1 - 2^{n-1}))\right) \text{ for } x \in I_n$$

For all n, consider the composition $p_n \circ \Psi$. This is a map $[0,1]/\sim \to C_n$ that is constant $(0,0)$ on $[0,1] \sim -I_n$, and defined (up to rescaling) as the parametrization ψ_n on I_n. It is a standard exercise of point-set topology to show that $p_n \circ \Psi$ is a homotopy equivalence $S^1 \to C_n$. Let a be a generator of $H_1(S^1)$ and c_n be a generator of $H_1(C_n)$. Since $p_n \circ \Psi$ is a homotopy equivalence, we have

$$(p_n \circ \Psi)_*(a) = \pm c_n.$$

This implies that $\Psi_*(a)$ is non-trivial in $H_1(X)$. Suppose that $\Psi_*(a)$ was in the image of the map $\bigoplus(i_n)_*\colon \bigoplus H_1(C_n) \to H_1(X)$ and let $B \in \bigoplus H_1(C_n)$ be a preimage. By definition of direct sum, B can be written as a finite sum $B = \sum_{j=1}^{k} b_{i_j}$, with $b_{i_j} \in H_1(C_{i_j})$ for some i_j. Let $N = \max\{i_j\} + 1$. Then $(p_N)_*(B) = 0$. Since $(p_N)_*(a) = \pm c_N \neq 0$, we obtain that B could not be a preimage of a.