Autumn Semester 2019

Introduction to Algebraic Number Theory

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Exercise Sheet 1

Exercise 1.1. Consider the ring $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$.

(a) Show that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain with respect to $N(\alpha) = |\alpha|^2 = a^2 + 2b^2$, where $\alpha = a + b\sqrt{-2}$;

(b) Show that $\mathbb{Z}[\sqrt{-2}]^\times = \{\pm 1\}$;

(c) Show that if y is an odd integer, then $\gcd(y - \sqrt{-2}, y + \sqrt{-2}) = 1$.

Exercise 1.2. Let $\omega = e^{\frac{2\pi i}{3}} = \frac{-1+\sqrt{-3}}{2}$, and consider the ring $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$.

(a) Show that $\mathbb{Z}[\omega]$ is Euclidean with respect to $N(a + b\omega) = a^2 - ab + b^2$;

(b) Show that $\mathbb{Z}[\omega]^\times = \{\pm 1, \pm\omega, \pm\omega^2\}$;

(c) Using the fact that $\mathbb{Z}[\omega]$ is a UFD show that for all primes $p > 3$ we have

$$p = a^2 - ab + b^2 \ (a, b \in \mathbb{Z}) \quad \Leftrightarrow \quad p \equiv 1 \pmod{3}.$$

Exercise 1.3. Find all integral solutions of

(a) $y^2 = x^3 + 7$;

(b) $y^2 = x^3 - 6$;

(c) $y^2 = x^3 - 4$.

Hint: in (a) and (b) use quadratic reciprocity.

Exercise 1.4*. Find all integral solutions of $y^2 + 2 = 3^n$.