Autumn Semester 2019

Introduction to Algebraic Number Theory

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Exercise Sheet 10

Exercise 10.1. Describe all integer solutions of $x^2 - 2y^2 = 7$.

Exercise 10.2. Let p be an odd prime, ζ_p a primitive p-th root of unity, and $K = \mathbb{Q}(\zeta_p)$.

(a) Show that the set of roots of unity in K is $\mu_K = \{\pm\zeta_p^j \mid j = 0, \ldots, p-1\}$;

(b) Show that $u_j = \frac{\zeta_p^j - 1}{\zeta_p - 1}$ is a unit in $\mathcal{O}_K$ for $j = 1, \ldots, p-1$.

Exercise 10.3. Let $K = \mathbb{Q}(\sqrt{5}, \sqrt{-2})$.

(a) Show that $\mathcal{O}_K = \mathbb{Z}[\frac{\sqrt{5}+1}{2}, \sqrt{-2}]$;

(b) Show that the only roots of unity in $\mathcal{O}_K$ are ± 1;

(c) Show that $\frac{\sqrt{5}+1}{2}$ is a fundamental unit in K, i.e., $\mathcal{O}_K^\times = \{\pm(\frac{\sqrt{5}+1}{2})^n \mid n \in \mathbb{Z}\}$.

(*Hint: in (a) use Exercise 4.4; in (b) and (c) use the norm maps $N_{K/F}$ for quadratic subfields F.*)

Exercise 10.4. Let $K = \mathbb{Q}(\sqrt[3]{2})$. Show that $\mathcal{O}_K^\times = \{\pm(\sqrt[3]{2} - 1)^n \mid n \in \mathbb{Z}\}$.

(*Hint: Use the embedding $i\colon \mathcal{O}_K \to \mathbb{R} \times \mathbb{C}$ and find a bounded region $B \subseteq \mathbb{R} \times \mathbb{C}$ such that for any unit $\alpha \in \mathcal{O}_K^\times$, $i(\pm\alpha(\sqrt[3]{2} - 1)^n) \in B$ for some n.*)