Autumn Semester 2019

Introduction to Algebraic Number Theory

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Exercise Sheet 11

Exercise 11.1. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Recall from the lectures that $\mathcal{O}_K = \mathbb{Z}[\gamma]$, where $\gamma = \frac{\sqrt{2}+\sqrt{6}}{2}$. Show that no prime is inert in K, i.e., that (p) is not a prime ideal of $\mathcal{O}_K$ for any prime p.

(*Hint: Use the fact that for any p one of* 2, 3, *or* 6 *is a quadratic residue mod p.*)

Exercise 11.2. Let $K = \mathbb{Q}(\alpha)$, where α is a root of $p(x) = x^3 - 10x^2 + 19x - 2$.

(a) Show that $D_K = 43^2$;

(b) Check that $\alpha_2 = \frac{-\alpha^2+7\alpha+4}{2}$ and $\alpha_3 = \frac{\alpha^2-9\alpha+16}{2}$ are roots of $p(x)$, and show that K is a Galois extension;

(c) Show that $(2) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ for distinct prime ideals $\mathfrak{p}_i$;

(d) Conclude that $\mathcal{O}_K$ is not monogenic, i.e., $\mathcal{O}_K \neq \mathbb{Z}[\theta]$ for any $\theta \in K$.

Exercise 11.3. Let $K = \mathbb{Q}(\sqrt{-D})$, where $D \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$. Show that the class number of K is equal to 1.

(*Hint: Cases $D = 1, 2, 3, 11, 19$ where previously considered in Ex. 1.1, Ex. 1.2, Ex. 8.1b. In remaining cases compute the Minkowski bound and factorize small primes into prime ideals.*)

Exercise 11.4. Let $K = \mathbb{Q}(\alpha)$, where α is an algebraic integer with minimal polynomial $q(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Show that if $q(x)$ is Eisenstein at prime p (i.e., $p||a_0$ and $p|a_j$, $j = 1, \ldots, n-1$), then $p \nmid [\mathcal{O}_K \colon \mathbb{Z}[\alpha]]$.