Autumn Semester 2019

# Introduction to Algebraic Number Theory

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## Exercise Sheet 12

**Exercise 12.1.** Show that the ring of integers of $K = \mathbb{Q}(\zeta_{23})$ is not a PID:

(a) Show that $K$ contains a subfield $F$ isomorphic to $\mathbb{Q}(\sqrt{-23})$;

(b) Show that 47 splits completely in $\mathcal{O}_K$;

(c) Assume that some prime ideal above 47 is of the form $(x)$ for $x \in \mathcal{O}_K$. Show that $y = N_{K/F}(x) \in \mathcal{O}_F$ has norm 47, and obtain a contradiction.

(*Hint: in part (a) use Gauss sums.*)

**Exercise 12.2.** Let $p$ be an odd prime and let $K = \mathbb{Q}(\zeta_p) \subset \mathbb{C}$, where $\zeta_p = e^{2\pi i/p}$ is a primitive $p$-th root of unity. Show that any unit $u \in \mathcal{O}_K^\times$ can be written as $u = r\zeta_p^n$ for some $r \in \mathbb{R} \cap \mathcal{O}_K^\times$ and $n \in \{0, \ldots, p-1\}$.

**Exercise 12.3.** We call a prime $p$ regular if $p$ does not divide the class number of $\mathbb{Q}(\zeta_p)$. Show that if $p \geq 5$ is regular and $x^p + y^p + z^p = 0$ for some $x, y, z \in \mathbb{Z}$, then $p|xyz$ as follows:

Assume that $x$, $y$, and $z$ are relatively prime and $p \nmid xyz$.

(a) Show that the ideals $(x + \zeta_p^j y)$ are relatively prime for $j = 0, \ldots, p-1$;

(b) Show that $x + \zeta_p y = r\zeta_p^n \alpha^p$ for some $\alpha \in \mathbb{Z}[\zeta_p]$, $r \in \mathbb{Z}[\zeta_p]^\times \cap \mathbb{R}$ and $n \in \{0, \ldots, p-1\}$;

(c) Show that $\alpha^p \equiv a \pmod p$ for some integer $a$;

(d) Using parts (b) and (c) show that

$$\gamma = \zeta_p^n x + \zeta_p^{n-1} y - \zeta_p^{-n} x - \zeta_p^{-n+1} y \equiv 0 \pmod p$$

(e) Obtain contradiction using part (d).