Autumn Semester 2019

Introduction to Algebraic Number Theory

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Exercise Sheet 2

Exercise 2.1. Let $A = \mathbb{Z} + \mathbb{Z}\sqrt{-3}$ and $K = \mathbb{Q}(\sqrt{-3})$. Show that the integral closure of A in K is $\mathbb{Z} + \mathbb{Z}(\frac{1+\sqrt{-3}}{2})$.

Exercise 2.2. Show that the following numbers are algebraic integers:

(a) $\alpha = \frac{\sqrt{3}+\sqrt{7}}{2}$;

(b) $\beta = \frac{\alpha^2-\alpha}{2}$, where α is a root of $x^3 + x - 6$. (To find a polynomial equation that β satisfies, write down the matrix of multiplication by β in $\mathbb{Q}(\alpha)$.)

Exercise 2.3. Let p be a prime, and let ζ_p be a primitive p-th root of unity. Show that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\zeta_p)$ is $\mathbb{Z}[\zeta_p]$ as follows. Let $\alpha = a_0 + a_1\zeta_p + \cdots + a_{p-2}\zeta_p^{p-2}$.

(a) Using the Galois action show that if α is an algebraic integer, then $pa_j \in \mathbb{Z}$;

(b) Show that $\frac{p}{(1-\zeta_p)^l}$ is an algebraic integer for $0 \leq l \leq (p-1)$;

(c) Show that α can be written as

$$\alpha = \frac{m_0 + m_1(1-\zeta_p) + \cdots + m_{p-2}(1-\zeta_p)^{p-2}}{p},$$

where $m_i \in \mathbb{Z}$, and, assuming that some m_j is not divisible by p, obtain a contradiction using part (b).

Exercise 2.4 (Gauss sums). Let p be an odd prime, and let ζ_p be a primitive p-th root of unity. For $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ define

$$\tau(a) = \sum_{x\in(\mathbb{Z}/p\mathbb{Z})^\times} \left(\frac{x}{p}\right) \zeta_p^{ax},$$

where $(\frac{a}{p})$ is the Legendre symbol.

(a) Show that $\tau(a) = (\frac{a}{p})\tau(1)$;

(b) Show that $\tau(1)^2 = (-1)^{\frac{p-1}{2}} p$;

(c) Let $K/\mathbb{Q}$ be a quadratic extension. Prove that there exists a root of unity ζ such that $K \subseteq \mathbb{Q}(\zeta)$.

Hint: in (c) note that $\mathbb{Q}(\sqrt{d}) \subset \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots)$, *and use the result of part (b).*