Autumn Semester 2019

Introduction to Algebraic Number Theory

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Exercise Sheet 3

Exercise 3.1. Let $K = \mathbb{Q}(\alpha)$, where $\alpha = \sqrt[4]{2}$, and let $\mathrm{Tr} = \mathrm{Tr}_{K/\mathbb{Q}}$.

(a) Show that $\mathrm{Tr}(a + b\alpha + c\alpha^2 + d\alpha^3) = 4a$ for $a, b, c, d \in \mathbb{Q}$;

(b) Use part (a) to show that $\sqrt{3} \notin K$.

Hint: in (b) assume that $\sqrt{3} = a + b\alpha + c\alpha^2 + d\alpha^3 \in K$, *where* $\alpha = \sqrt[4]{2}$, *and obtain contradiction by consecutively computing* $\mathrm{Tr}(\sqrt{3}/\alpha^j)$, $j = 0, 1, 2, 3$.

Exercise 3.2. Let $K/\mathbb{Q}$ be an algebraic extension of degree n, and let $\alpha_1, \dots, \alpha_n \in \mathcal{O}_K$.

(a) Let $\sigma_1, \dots, \sigma_n$ be the complex embeddings of K and define

$$P = \sum_{\substack{\pi \in S_n \\ \mathrm{sgn}(\pi)=1}} \prod_{j=1}^{n} \sigma_{\pi(j)}(\alpha_j)\,,$$

$$N = \sum_{\substack{\pi \in S_n \\ \mathrm{sgn}(\pi)=-1}} \prod_{j=1}^{n} \sigma_{\pi(j)}(\alpha_j)\,.$$

Show that $P + N$ and PN are integers.

(b) Use part (a) to show that the discriminant $\mathrm{d}(\alpha_1, \dots, \alpha_n)$ is congruent to 0 or 1 modulo 4.

(c) Let $\sigma_1, \dots, \sigma_{r_1}, \sigma_{r_1+1}, \overline{\sigma}_{r_1+1}, \dots, \sigma_{r_1+r_2}, \overline{\sigma}_{r_1+r_2}$, $n = r_1 + 2r_2$ be the complex embeddings of K, where $\sigma_i(K) \subset \mathbb{R}$, $i = 1, \dots, r_1$, and $\sigma_i(K) \not\subset \mathbb{R}$, $i = r_1 + 1, \dots, r_1 + r_2$. Assuming that $\mathrm{d}(\alpha_1, \dots, \alpha_n) \neq 0$ show that its sign is $(-1)^{r_2}$.

Exercise 3.3. Let $K = \mathbb{Q}(\alpha)$, where $\alpha^3 - \alpha^2 - 2\alpha - 8 = 0$. Recall that $\mathcal{O}_K$ is a free $\mathbb{Z}$-module spanned by $\{\omega_1, \omega_2, \omega_3\}$ for some $\omega_i \in \mathcal{O}_K$.

(a) Compute the discriminant $\mathrm{d}(1, \alpha, \frac{\alpha^2-\alpha}{2})$;

(b) Show that $\mathcal{O}_K$ is the integral span of $\{1, \alpha, \frac{\alpha^2-\alpha}{2}\}$;

(c) Show that $\mathcal{O}_K$ **does not** have the form $\mathbb{Z}[\gamma]$ for any $\gamma \in \mathcal{O}_K$.

Hint: in (c) compute the transition matrix from $\{1, \gamma, \gamma^2\}$ *to* $\{1, \alpha, \frac{\alpha^2-\alpha}{2}\}$.

Exercise 3.4*. Let $K = \mathbb{Q}(\sqrt{-2}, \sqrt{-5})$.

(a) Show that $\mathcal{O}_K$ is the integral span of $\{1, \sqrt{-2}, \sqrt{-5}, \frac{\sqrt{-2}+\sqrt{10}}{2}\}$;

(b) Show that $\mathcal{O}_K$ **does not** have the form $\mathbb{Z}[\gamma]$ for any $\gamma \in \mathcal{O}_K$.