Autumn Semester 2019

Introduction to Algebraic Number Theory

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Exercise Sheet 4

Exercise 4.1. A ring R is called Noetherian if every ideal of R is finitely generated. Show that the following conditions are equivalent:

(i) R is Noetherian;

(ii) every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ stabilizes (i.e. there exists n such that $I_n = I_{n+1} = \dots$);

(iii) Every non-empty set of ideals has a maximal element.

Exercise 4.2. Let $K = \mathbb{Q}(\alpha)$, where $\alpha^3 - \alpha - 2 = 0$. (We assume that it is known that the polynomial $x^3 - x - 2$ is irreducible.)

(a) Compute the discriminant of $\{1, \alpha, \alpha^2\}$;

(b) Let $\Lambda = \mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\alpha^2 \subseteq \mathcal{O}_K$. Use the discriminant to show that $|\mathcal{O}_K/\Lambda| \leq 2$;

(c) Prove that $\frac{\alpha}{2}$, $\frac{\alpha^2}{2}$, and $\frac{\alpha^2+\alpha}{2}$ are not algebraic integers. Conclude that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

Exercise 4.3. A cubic extension $K/\mathbb{Q}$ (i.e. $[K : \mathbb{Q}] = 3$) is called a pure cubic field if it is of the form $\mathbb{Q}(\sqrt[3]{n})$ for some integer n that is not a cube.

(a) Show that the discriminant of a pure cubic field is equal to $-3d^2$ for some $d \in \mathbb{Z}$;

(b) Show that $\mathbb{Q}(\theta)$, where $\theta^3 - 3\theta + 4 = 0$ is not a pure cubic field.

Exercise 4.4. Let $K_1 = \mathbb{Q}(\gamma_1)$ and $K_2 = \mathbb{Q}(\gamma_2)$ be algebraic extensions of degrees n_1 and n_2 respectively, such that $K = \mathbb{Q}(\gamma_1, \gamma_2)$ has degree $n_1 n_2$ over $\mathbb{Q}$. Let $\{\alpha_1, \dots, \alpha_{n_1}\}$ and $\{\beta_1, \dots, \beta_{n_2}\}$ be integral bases of K_1 and K_2 respectively, of discriminants D_1 and D_2.

(a) Show that $\{\alpha_i \beta_j\}_{i,j}$ form a basis for K over $\mathbb{Q}$;

(b) Fix some embedding of K into $\mathbb{C}$. Show that there are exactly n_1 embeddings $\varphi_i \colon K \to \mathbb{C}$ that restrict to identity on K_2, and that there are exactly n_2 embeddings $\psi_j \colon K \to \mathbb{C}$ that restrict to identity on K_1;

(c) Let $\omega = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} u_{ij} \alpha_i \beta_j \in \mathcal{O}_K$, where $u_{ij} \in \mathbb{Q}$. Show that $D_2 u_{ij} \in \mathbb{Z}$ for all i, j;

(d) Show that if D_1 and D_2 are coprime, then $\{\alpha_i \beta_j\}_{i,j}$ forms an integral basis for K;

(e) Show that the discriminant of $\{\alpha_i \beta_j\}_{i,j}$ is $D_1^{n_2} D_2^{n_1}$ *(Hint: recall the Kronecker product of matrices)*;

(f) Using (d) and (e) write down an integral basis for the quartic field $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$ and find its discriminant.