Autumn Semester 2019

Introduction to Algebraic Number Theory

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Exercise Sheet 5

Exercise 5.1. Find the discriminant of $K = \mathbb{Q}(\zeta_p)$, where ζ_p is a primitive p-th root of unity and p is an odd prime.

(*Hint: recall from Exercise 2.3 that* $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$.)

Exercise 5.2. Recall that a quadratic field $K/\mathbb{Q}$ is uniquely determined by its discriminant. In this exercise we will show that this is no longer true for (pure) cubic fields. Let $K_1 = \mathbb{Q}(\sqrt[3]{6})$ and $K_2 = \mathbb{Q}(\sqrt[3]{12})$.

(a) Show that $\mathcal{O}_{K_1}$ is spanned by $\{1, \sqrt[3]{6}, \sqrt[3]{36}\}$;

(b) Show that $\mathcal{O}_{K_2}$ is spanned by $\{1, \sqrt[3]{12}, \sqrt[3]{18}\}$;

(c) Show that $D_{K_1} = D_{K_2}$, but K_1 and K_2 are not isomorphic.

Exercise 5.3. Let $K = \mathbb{Q}(\sqrt{6})$.

(a) Show that $\mathfrak{p} = (2, \sqrt{6})$ is a prime ideal in $\mathcal{O}_K$;

(b) Find the generators of $\mathfrak{p}^{-1} = \{a \in \mathcal{O}_K \mid a\mathfrak{p} \subseteq \mathcal{O}_K\}$.

(c) Show that $\mathfrak{p}$ is principal and find its generator.

Exercise 5.4. Let $D \equiv 1 \pmod 4$ be a squarefree number, $D > 1$, and let $K = \mathbb{Q}(\sqrt{-D})$. Show that $\mathfrak{p} = (2, 1 + \sqrt{-D})$ is not a principal ideal of $\mathcal{O}_K$.

Exercise 5.5. Show that the ring of integers $\mathcal{O}_K$ is a unique factorization domain if and only if it is a principal ideal domain.

(*Hint: Use unique factorization of ideals in* $\mathcal{O}_K$.)