Autumn Semester 2019

Introduction to Algebraic Number Theory

Lecturer: Prof. Dr. Özlem Imamoglu
Coordinator: Dr. Danylo Radchenko

Exercise Sheet 6

Exercise 6.1. Let $K = \mathbb{Q}(\sqrt{-30})$ and define $\mathfrak{a} = (2, \sqrt{-30})$ and $\mathfrak{b} = (3, \sqrt{-30})$.

(a) Find the norms of $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{a}\mathfrak{b}$;

(b) Show that $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{a}\mathfrak{b}$ are not principal and represent different elements of the ideal class group.

(*Hint: show that* $\mathfrak{a}^2 = (2)$ *and* $\mathfrak{b}^2 = (3)$.)

Exercise 6.2. Let $K = \mathbb{Q}(\sqrt{5})$ and let $A = \mathbb{Z}[\sqrt{5}]$ (note that $A \neq \mathcal{O}_K$).

(a) Prove that the ideal $\mathfrak{p} = (2, 1 + \sqrt{5})$ in A is maximal, and $|A/\mathfrak{p}| = 2$.

(b) Prove that $\mathfrak{p}^2 = 2\mathfrak{p}$ and $\mathfrak{p}^{-1} = \frac{1}{2}\mathfrak{p}$, so that $\mathfrak{p}\mathfrak{p}^{-1} = \mathfrak{p}$;

(c) Let $\mathfrak{a} = (2)$. Show that $\mathfrak{a} \subseteq \mathfrak{p}$, but there is no ideal $\mathfrak{b}$ such that $\mathfrak{a} = \mathfrak{p}\mathfrak{b}$ (i.e, $\mathfrak{p} \nmid \mathfrak{a}$).

(*Hint: in (c) suppose that* $\mathfrak{a} = \mathfrak{p}\mathfrak{b}$ *for some integral ideal* $\mathfrak{b}$ *and use (b) to show that* $\mathfrak{a} = \mathfrak{p}$.)

Exercise 6.3. Let K be a number field and assume that $|Cl_K| = 2$.

(a) Let $a \in \mathcal{O}_K$ be an irreducible element that is not prime. Show that $(a) = \mathfrak{p}_1\mathfrak{p}_2$ for some non-principal prime ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$;

(b) Show that any two factorizations of $a \in \mathcal{O}_K$ into irreducibles have the same number of irreducible factors.

Exercise 6.4. In this exercise we give a different definition of the ideal class group. Let K be a number field, $A = \mathcal{O}_K$, and let $\mathcal{I}$ be the set of all integral ideals of A. We say that $\mathfrak{a} \sim \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ if there are non-zero elements $x, y \in A$ such that $x\mathfrak{a} = y\mathfrak{b}$.

(a) Show that $\sim$ is an equivalence relation on $\mathcal{I}$;

(b) Show that if $\mathfrak{a} \sim \mathfrak{a}'$ and $\mathfrak{b} \sim \mathfrak{b}'$, then $\mathfrak{a}\mathfrak{b} \sim \mathfrak{a}'\mathfrak{b}'$, and show that there is $\widetilde{\mathfrak{a}} \in \mathcal{I}$ such that $\mathfrak{a}\widetilde{\mathfrak{a}} \sim A$, so that $\mathcal{I}/\sim$ is a group whose identity is the equivalence class of A;

(c) Show that the group $\mathcal{I}/\sim$ is isomorphic to the class group defined in terms of fractional ideals.