Autumn Semester 2019

Introduction to Algebraic Number Theory

Lecturer: Prof. Dr. Özlem Imamoglu
Coordinator: Dr. Danylo Radchenko

Solutions to Exercise Sheet 7

Exercise 7.1. Show that $X = \mathbb{Z} + \mathbb{Z}\sqrt{2}$ is a dense subset of $\mathbb{R}$.

Solution. Since $\sqrt{2}$ is irrational, $a + b\sqrt{2} = 0$ for $a, b \in \mathbb{Z}$ if and only if $a = b = 0$. Consider $x_n = n\sqrt{2} - \lfloor n\sqrt{2} \rfloor \in X \cap [0, 1]$ for $n \geq 1$. If we partition $[0, 1]$ into N segments of length $1/N$, we see by pigeonhole principle that for some $m \neq n$ one has $|x_m - x_n| < 1/N$. Thus for any $\varepsilon > 0$ there exist $n \neq m$ such that $|x_n - x_m| < \varepsilon$. Note that $x_n - x_m \in X$.

Therefore X contains arbitrarily small non-zero numbers. But if (t_1, t_2) is any interval of length $> \varepsilon$, and $x \in X$ satisfies $|x| < \varepsilon$, then (t_1, t_2) contains some element of the form mx, $m \in \mathbb{Z}$, i.e., $(t_1, t_2) \cap X \neq \emptyset$. Therefore, X is a dense subset of $\mathbb{R}$.

Exercise 7.2. Let Λ be a complete lattice in $\mathbb{R}^n$ and denote by $|\Lambda|$ the volume of its fundamental region.

(a) Find a convex centrally-symmetric set $K \subset \mathbb{R}^n$ such that $\operatorname{vol}(K) = 2^n|\Lambda|$, but $K \cap \Lambda = \{0\}$;

(b) Show that if K is a compact convex centrally-symmetric set and $\operatorname{vol}(K) \geq 2^n|\Lambda|$, then there exists $0 \neq \nu \in K \cap \Lambda$. (Recall that Minkowski's theorem requires that $\operatorname{vol}(K) > 2^n|\Lambda|$, but does not require compactness.)

Solution. (a) If $\Lambda = \mathbb{Z}^n$ then $|\Lambda| = 1$ and $K = (-1, 1)^n$ is a convex centrally-symmetric set that satisfies $\operatorname{vol}(K) = 2^n$ and $K \cap \Lambda = \{0\}$. In general, if A is an invertible linear transformation such that $A(\mathbb{Z}^n) = \Lambda$ (for example, one can take any basis of Λ as the set of columns of A), then $K = A((-1, 1)^n)$ satisfies $\operatorname{vol}(K) = 2^n|\Lambda|$ and $K \cap \Lambda = A((-1, 1)^n \cap \mathbb{Z}^n) = \{0\}$.

(b) For $m \geq 1$ let $K_m = (1 + 1/m)K$. Since $\operatorname{vol}(K_m) = (1 + 1/m)^n \operatorname{vol}(K) > 2^n|\Lambda|$, by Minkowski's theorem we get that there exists a non-zero vector $\nu_m \in K_m \cap \Lambda$. Since the sets $K_m \cap \Lambda$ are all contained in a finite set $\subseteq (2K) \cap \Lambda$, this means that there exists a non-zero vector $\nu \in \Lambda$ that belongs to infinitely many K_m's.

On the other hand, the sequence of sets $\{K_m\}_{m \geq 1}$ is decreasing with respect to inclusion, thus $\nu \in \bigcap_{m \geq 1} K_m$. By definition of K_m this means that $\frac{m}{m+1}\nu \in K$ for all $m \geq 1$. Since K is compact, it is closed, so that the limit of $\frac{m}{m+1}\nu$ as $m \to \infty$ also belongs to K. Since this limit is $\nu \in \mathbb{Z}^n \smallsetminus \{0\}$, we have proved the claim.

Exercise 7.3. Let $K \subset \mathbb{R}^n \times \mathbb{C}^m$ be given by

$$K = \{(x_1, \ldots, x_n, z_1, \ldots, z_m) \mid \sum_{i=1}^{n} |x_i| + \sum_{j=1}^{m} |z_j| \leq 1\}.$$

Show that $\operatorname{vol}(K) = \frac{2^n (2\pi)^m}{(n+2m)!}$. (*Hint: use induction.*)

Solution. To emphasize the dependence on n and m, denote K by $K_{n,m}$, and let $V_{n,m}$ be its volume. Since $\mathbb{R}^n \times \mathbb{C}^m$ is $(n+2m)$-dimensional, the volume of $tK_{n,m}$ is given by $V_{n,m}t^{n+2m}$.

Assume that $m \geq 1$. Fixing the value of z_m to be z with $|z| \leq 1$, we see that $K_{n,m} \cap \{z_m = z\} = (1-|z|)K_{n,m-1}$. Thus we can compute the volume of $K_{n,m}$ by integrating over the variable z_m:

$$V_{n,m} \;=\; \int_{|z|\leq 1} (1-|z|)^{n+2(m-1)} V_{n,m-1} dxdy \;=\; V_{n,m-1} \int_{|z|\leq 1} (1-|z|)^{n+2(m-1)} dxdy\,,$$

where $z = x+iy \in \mathbb{C}$. We compute the last integral by changing to polar coordinates:

$$\begin{aligned}\int_{|z|\leq 1} (1-|z|)^{n+2(m-1)} dxdy \;&=\; \int_0^1 (1-t)^{n+2m-2} 2\pi t dt \;=\; 2\pi \int_0^1 t^{n+2m-2}(1-t)dt \\ &=\; 2\pi\Big(\frac{1}{n+2m-1} - \frac{1}{n+2m}\Big) \;=\; \frac{2\pi}{(n+2m-1)(n+2m)}\,.\end{aligned}$$

Repeatedly applying this we see that

$$V_{n,m} \;=\; \frac{(2\pi)^m n!}{(n+2m)!} V_{n,0}\,.$$

Similarly, we compute $V_{n,0}$ by integrating the $(n-1)$-dimensional volumes of slices for $x_n = x \in [-1,1]$:

$$V_{n,0} \;=\; \int_{-1}^1 (1-|x|)^{n-1} V_{n-1,0} dx \;=\; 2V_{n-1,0}\int_0^1 (1-t)^{n-1}dt \;=\; \frac{2}{n}V_{n-1,0}\,.$$

Since $V_{1,0} = 2$, we get from this recursion that $V_{n,0} = \frac{2^n}{n!}$, and hence

$$V_{n,m} = \frac{(2\pi)^m n!}{(n+2m)!}\frac{2^n}{n!} \;=\; \frac{2^n(2\pi)^m}{(n+2m)!}\,.$$

Exercise 7.4. Let $l_i(x_1,\ldots,x_n) = \sum_{j=1}^n a_{ij}x_j$, $i = 1,\ldots,n$, be real linear forms such that $D = |\det(a_{ij})| \neq 0$.

(a) Let $c_1,\ldots,c_n > 0$ be such that $c_1\ldots c_n > D$. Show that there exists a vector $(m_1,\ldots,m_n) \in \mathbb{Z}^n \smallsetminus \{0\}$ such that

$$|l_i(m_1,\ldots,m_n)| < c_i\,, \qquad i = 1,\ldots,n\,.$$

(b) Show that there exists a vector $(m_1,\ldots,m_n) \in \mathbb{Z}^n \smallsetminus \{0\}$ such that

$$\sum_{i=1}^n |l_i(m_1,\ldots,m_n)| \leq (n!D)^{1/n}\,.$$

Solution. Let $A = (a_{ij})_{i,j}$. Note that by assumption A is invertible.

(a) Let $K = \{x \in \mathbb{R}^n \mid |l_i(x)| < c_i\}$. Then $K = A^{-1}(Q)$, where $Q = (-c_1,c_1)\times \cdots \times (-c_n,c_n)$. Clearly, K is centrally symmetric, and from $K = A^{-1}(Q)$ we see that

$\mathrm{vol}(K) = 2^n D^{-1} c_1 \ldots c_n > 2^n$. Thus by Minkowski's Theorem there exists a non-zero integer vector $(m_1, \ldots, m_n) \in K \cap \mathbb{Z}^n$.

(b) Denote $t = (n!D)^{1/n}$ and let $K = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n |l_i(x)| \le t\}$. Then $K = tA^{-1}(L)$, where L is the centrally symmetric convex body from Ex. 7.3 for $m = 0$. Thus $\mathrm{vol}(K) = t^n D^{-1} 2^n / n! = 2^n$. Since K is compact, we get the claim by applying the refined version of the Minkowski Theorem from Ex. 7.2 (b).

Exercise 7.5. Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be an irreducible polynomial with integral coefficients. Let $\alpha_1, \ldots, \alpha_n$ be the roots of p and let $\Delta = \prod_{i<j}(\alpha_i - \alpha_j)^2$ be its discriminant. Show that if all α_i are real numbers, then

$$|\Delta| \ge \left(\frac{n^n}{n!}\right)^2.$$

(*Hint: use 7.4 (b) with* $l_i(x_1, \ldots, x_n) = \sum_{j=1}^n x_j \alpha_i^{j-1}$)

Solution. Consider the linear forms $l_i(x_1, \ldots, x_n) = \sum_{j=1}^n x_j \alpha_i^{j-1}$. The determinant of the associated $n \times n$ matrix is equal to $\pm\sqrt{|\Delta|} \ne 0$. Note that for any non-zero integer vector $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ the product $\prod_{i=1}^n l_i(m_1, \ldots, m_n)$ is a non-zero integer, since it is the norm of a non-zero algebraic integer $m_1 + m_2\alpha_1 + \cdots + m_n\alpha_1^{n-1}$. From this, using the inequality between arithmetic and geometric means, we obtain

$$\sum_{i=1}^n |l_i(m_1, \ldots, m_n)| \ge n \prod_{i=1}^n |l_i(m_1, \ldots, m_n)|^{1/n} \ge n$$

for all $(m_1, \ldots, m_n) \in \mathbb{Z}^n \smallsetminus \{0\}$. Therefore, exercise Ex. 7.4 (b) implies that

$$(n!\sqrt{|\Delta|})^{1/n} \ge n\,,$$

which is equivalent to

$$|\Delta| \ge \left(\frac{n^n}{n!}\right)^2.$$