Autumn Semester 2019

Introduction to Algebraic Number Theory

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Exercise Sheet 7

Exercise 7.1. Show that $X = \mathbb{Z} + \mathbb{Z}\sqrt{2}$ is a dense subset of $\mathbb{R}$.

Exercise 7.2. Let Λ be a complete lattice in $\mathbb{R}^n$ and denote by $|\Lambda|$ the volume of its fundamental region.

(a) Find a convex centrally-symmetric set $K \subset \mathbb{R}^n$ such that $\mathrm{vol}(K) = 2^n|\Lambda|$, but $K \cap \Lambda = \{0\}$;

(b) Show that if K is a compact convex centrally-symmetric set and $\mathrm{vol}(K) \geq 2^n|\Lambda|$, then there exists $0 \neq \nu \in K \cap \Lambda$. (Recall that Minkowski's theorem requires that $\mathrm{vol}(K) > 2^n|\Lambda|$, but does not require compactness.)

Exercise 7.3. Let $K \subset \mathbb{R}^n \times \mathbb{C}^m$ be given by

$$K = \{(x_1, \ldots, x_n, z_1, \ldots, z_m) \mid \sum_{i=1}^{n} |x_i| + \sum_{j=1}^{m} |z_j| \leq 1\}.$$

Show that $\mathrm{vol}(K) = \frac{2^n(2\pi)^m}{(n+2m)!}$. *(Hint: use induction.)*

Exercise 7.4. Let $l_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} a_{ij}x_j$, $i = 1, \ldots, n$, be real linear forms such that $D = |\det(a_{ij})| \neq 0$.

(a) Let $c_1, \ldots, c_n > 0$ be such that $c_1 \ldots c_n > D$. Show that there exists a vector $(m_1, \ldots, m_n) \in \mathbb{Z}^n \setminus \{0\}$ such that

$$|l_i(m_1, \ldots, m_n)| < c_i\,, \qquad i = 1, \ldots, n\,.$$

(b) Show that there exists a vector $(m_1, \ldots, m_n) \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\sum_{i=1}^{n} |l_i(m_1, \ldots, m_n)| \leq (n!D)^{1/n}\,.$$

Exercise 7.5. Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be an irreducible polynomial with integral coefficients. Let $\alpha_1, \ldots, \alpha_n$ be the roots of p and let $\Delta = \prod_{i<j}(\alpha_i - \alpha_j)^2$ be its discriminant. Show that if all α_i are real numbers, then

$$|\Delta| \geq \left(\frac{n^n}{n!}\right)^2.$$

(Hint: use 7.4 (b) with $l_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j\alpha_i^{j-1}$*)*