Autumn Semester 2019

Introduction to Algebraic Number Theory

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Exercise Sheet 9

Exercise 9.1. Let $\{a_n\}_{n\geq 1}$ be a sequence of positive integers, let $a_0 \geq 0$ and define

$$\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}},$$

where the fraction p_n/q_n is written in lowest terms and $q_n > 0$. Denote $(p_{-2}, q_{-2}) = (0, 1)$ and $(p_{-1}, q_{-1}) = (1, 0)$. Let $x = [a_0; a_1, a_2, \dots] := \lim_{n\to\infty} p_n/q_n$.

(a) Show that $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$, and $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$ for all $n \geq 0$. Deduce from this that $\lim_{n\to\infty} p_n/q_n$ exists;

(b) Show that $\operatorname{sgn}(x - \frac{p_n}{q_n}) = (-1)^n$ and $\frac{1}{q_n(q_{n+1}+q_n)} < |x - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}$ for $n \geq 0$;

(c) Let $\alpha_m = [a_m; a_{m+1}, \dots]$. Show that $x = \frac{p_m \alpha_{m+1} + p_{m-1}}{q_m \alpha_{m+1} + q_{m-1}}$ and $\alpha_{m+1} = \frac{p_{m-1} - x q_{m-1}}{x q_m - p_m}$.

Exercise 9.2. Let α be a quadratic irrational. Show that the continued fraction representation $\alpha = [a_0; a_1, a_2, \dots]$ is eventually periodic, i.e., $a_n = a_{n+T}$ for all $n \geq N$ for some $T > 0$, $N \geq 0$.

Exercise 9.3. Let $\alpha = \frac{P+\sqrt{D}}{Q} > 1$ be a quadratic irrational such that $-1 < \alpha' < 0$, where $\alpha' = \frac{P-\sqrt{D}}{Q}$. Show that the continued fraction $\alpha = [a_0; a_1, \dots]$ is purely periodic, i.e., $a_n = a_{n+T}$ for all $n \geq 0$.

Exercise 9.4. Let $D > 0$ be a non-square integer, let $[a_0; a_1, a_2, \dots]$ be the continued fraction representation of $\sqrt{D}$, and denote $\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$. Let T be the period of $\{a_n\}_{n\geq 0}$. Show that

$$p_{mT-1}^2 - D q_{mT-1}^2 = (-1)^{mT}, \quad m \geq 1\,.$$

(*Hint: apply Exercise 9.3 to* $\sqrt{D} + \lfloor\sqrt{D}\rfloor$ *and then use Exercise 9.1(c).*)