Autumn Semester 2019

Introduction to Algebraic Number Theory

Lecturer: Prof. Dr. Özlem Imamoglu
Coordinator: Dr. Danylo Radchenko

Ferienserie

Exercise 1. Let K be a quadratic extension of $\mathbb{Q}$ and let D be its discriminant. Show that $\mathcal{O}_K = \mathbb{Z}[\alpha_D]$, where $\alpha_D = \frac{D+\sqrt{D}}{2}$.

Exercise 2. Show that $\mathbb{Z}[\sqrt{2}]$ is a principal ideal domain, and using this show that

$$\pm p = x^2 - 2y^2 \quad \Leftrightarrow \quad p \equiv 1, 7 \pmod 8$$

(here p is a prime and $\pm p = x^2 - 2y^2$ means that either p or $-p$ can be written as $x^2 - 2y^2$).

Exercise 3. Show that the class group of $K = \mathbb{Q}(\sqrt{-23})$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and find the representative ideals.

Exercise 4. Let $K = \mathbb{Q}(\sqrt[3]{7})$.

(a) Show that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{7}]$;

(b) Show that the class number of K is equal to 3.

Exercise 5. Let $K = \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3})$. Show that $\alpha = (1 + \sqrt[3]{2})/\sqrt[3]{3}$ is a unit in $\mathcal{O}_K$.

Exercise 6. Let $p \equiv 1 \pmod 4$ be a prime number, and consider the element $\varepsilon \in \mathbb{Q}(\zeta_p)$ defined by

$$\varepsilon \;=\; \prod_{a=1}^{p-1} (1 - \zeta_p^a)^{\left(\frac{a}{p}\right)},$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

(a) Show that ε is a unit;

(b) Show that ε belongs to the quadratic subfield $\mathbb{Q}(\sqrt{p})$ in $\mathbb{Q}(\zeta_p)$;

(c) Compute ε for $p = 5$.

Exercise 7. Let $K/\mathbb{Q}$ be a Galois extension such that a prime number p is inert in K (i.e. (p) is a prime ideal). Show that $\mathrm{Gal}(K/\mathbb{Q})$ is a cyclic group.

Exercise 8. Prove that for any $n > 1$ there are infinitely many prime numbers congruent to 1 modulo n.

(*Hint: Assuming that there are only finitely many, let P denote their product. Obtain contradiction by considering a prime p dividing $\Phi_n(knP)$ for some $k \in \mathbb{Z}$, where Φ_n is the n-th cyclotomic polynomial.*)