D-MATH
HS 2019
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Exercise Sheet 1

Commutative Algebra

Let A, B be commutative rings with identity and let k be a field.

1 Let A be an integral domain with a finite number of elements. Show that A is a field. Deduce that in a finite commutative ring A every prime ideal is maximal.

2 Let $\mathfrak{a}, \mathfrak{b} \subseteq A$ be ideals. Prove that:

a. $\sqrt{\mathfrak{a}} = \sqrt{\sqrt{\mathfrak{a}}}$;

b. $\sqrt{\mathfrak{a}\mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$;

c. $\sqrt{\mathfrak{a}} = (1) \iff \mathfrak{a} = (1)$;

d. if $\mathfrak{p} \subseteq A$ is a prime ideal, then $\sqrt{\mathfrak{p}^k} = \mathfrak{p}$ for all integers $k > 0$.

3 Let n be a positive integer. Find the unique positive integer m such that $\sqrt{n\mathbb{Z}} = m\mathbb{Z}$.

4 Consider the polynomial ring $A[X]$. Let $f = \sum_{i=0}^{n} a_i X^i \in A[X]$ be a polynomial. Prove that:

a. f is a unit in $A[X]$ if and only if a_0 is a unit in A and $a_1, \dots, a_n$ are nilpotent;

b. f is nilpotent if and only if $a_0, \dots, a_n$ are nilpotent;

c. f is a zero-divisor if and only if there exists $a \neq 0$ in A such that $af = 0$.

Furthermore, prove that in the ring $A[X]$, the Jacobson radical is equal to the nilradical.

5 Let S be a multiplicative subset of A. Prove that S is a multiplicative subset of $A[X]$, and prove that there exists an isomorphism of rings

$$S^{-1}(A[X]) \longrightarrow (S^{-1}A)[X].$$

6 Let $f : A \to B$ be a ring morphism.

a. Let R be the set of all ordered pairs $(a, b) \in A \times A$ such that $f(a) = f(b)$. Show that R, with termwise addition and multiplication, is a ring.

b. One says that f is a *monomorphism* if for any ring C and any pair (g, g') of ring morphisms from C to A such that $f \circ g = f \circ g'$, one has $g = g'$.

   Show that a ring morphism is a monomorphism if and only if it is injective.

c. One says that f is an *epimorphism* if for any ring C and any pair (g, g') of ring morphisms from B to C such that $g \circ f = g' \circ f$, one has $g = g'$.

   Show that a surjective ring morphism is an epimorphism. Show also that the inclusion morphism from $\mathbb{Z}$ into $\mathbb{Q}$ is an epimorphism.

7 Let $A = k[X_1, \ldots, X_n]$ be the ring of polynomials with coefficients in k in n indeterminates. One says that an ideal of A is monomial if it is generated by monomials.

a. Let $(M_\alpha)_{\alpha \in E}$ be a family of monomials and let I be the ideal they generate. Show that a monomial M belongs to I if and only if it is a multiple of one of the monomials M_α.

b. Let I be an ideal of A. Show that I is a monomial ideal if and only if, for any polynomial $P \in I$, each monomial of P belongs to I.

c. Let I and J be monomial ideals of A. Show that the ideals $I + J$, IJ, $I \cap J$, $I :_A J = \{a \in A : aJ \subseteq I\}$ and $\sqrt{I}$ are again monomial ideals. Given monomials which generate I and J, explicit monomials which generate those ideals.

8 Let I be an ideal of A and let S be any subset of A. The conductor of S in I is defined by the formula

$$(I : S) = \{a \in A : as \in I \ \forall s \in S\}.$$

Show that it is an ideal of A, more precisely, the largest ideal J of A such that $JS \subseteq I$.