D-MATH
HS 2019
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Exercise Sheet 10

Commutative Algebra

Let $A \neq 0$ be a commutative ring.

1 Assume that A is noetherian and let $\mathfrak{p}$ be a prime ideal of A; let $S \subseteq \operatorname{Spec} A$ be a finite set of prime ideals such that for all $\mathfrak{p}' \in S$, $\mathfrak{p}' \not\supseteq \mathfrak{p}$. Show that if there exists a chain of primes

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{d-1} \subset \mathfrak{p},$$

then there exists such a chain with $\mathfrak{p}_1$ not contained in any ideal in S.
Hint: start with the case $d = 2$.

2 Assume that A is noetherian and let $\mathfrak{a}$ be an ideal of A, $\mathfrak{a} = (a_1, \ldots, a_n)$. Then for every prime ideal $\mathfrak{p} \supseteq \mathfrak{a}$ one has

$$\operatorname{ht}(\mathfrak{p}/\mathfrak{a}) \leq \operatorname{ht}(\mathfrak{p}) \leq \operatorname{ht}(\mathfrak{p}/\mathfrak{a}) + n.$$

3 Let A be a noetherian and local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. By Nakayama's Lemma, every minimal set of generators of $\mathfrak{m}$ has the same order, say $\mu(\mathfrak{m})$, which is equal to $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$. On the other hand, by the Hauptidealsatz, $\operatorname{ht}(\mathfrak{m}) \leq \mu(m)$. When the equality holds, A is said to be **regular**, i.e.

$$\dim A = \operatorname{ht}(\mathfrak{m}) = \dim_k(\mathfrak{m}/\mathfrak{m}^2).$$

a. Assume $\dim A = 0$. Then show that A is regular if and only if A is a field.

b. Show that if A is regular then $A/(a)$ is regular for all $a \in \mathfrak{m} - \mathfrak{m}^2$ and
$$\dim A = \dim(A/(a)) + 1.$$
Hint: use exercise 2.

c. Show that $A = \mathbb{Q}[X]/(X^2)$ is not regular and that every PID is regular.

d. Show that if A is regular, then A is an integral domain.
Hint: proceed by induction on $\dim A$ and consider $A/(a)$ with $a \in \mathfrak{m} - \mathfrak{m}^2$. Prove that A_a is UFD using the fact that a noetherian ring is UFD if and only if the primes of height 1 are principal.

e. Let $B = \mathbb{Q}[X,Y]/(f)$, where $f \in \mathbb{Q}[X,Y]$ is irreducible. Let $P = (X-x, Y-y) \subseteq \mathbb{Q}[X,Y]$ with $x, y \in \mathbb{Q}$ such that $f(x,y) = 0$. Let $A = B_P$; show that A is regular if and only if $(\partial_X f_{|_{(x,y)}}, \partial_Y f_{|_{(x,y)}}) \neq (0,0)$.

4 Show that $M = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ as $\mathbb{Z}$-module is artinian but not noetherian.