D-MATH
HS 2019
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Exercise Sheet 11

Commutative Algebra

1 Find a Noether normalization of the $\mathbb{C}$-algebra

$$B = \mathbb{C}[X, Y, Z]/(XY + Z^2, X^2Y - XY^3 + Z^4 - 1),$$

that is, algebraically independent elements $f_1, \dots, f_d$ in B such that B is a finitely generated module over the polynomial ring $\mathbb{C}[f_1, \dots, f_d]$. Also compute the Krull dimension of B.

2 Let $A := \mathbb{C}[X_{11}, X_{12}, X_{21}, X_{22}, Y_{11}, Y_{12}, Y_{21}, Y_{22}]$ and let $I \subseteq A$ be the ideal generated by the entries of the matrix XY, with

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \qquad Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

Write $V(I)$ as intersection of sets $V(\wp_i)$, with $\wp_i$ prime ideals in A. The $V(\wp_i)$ are the *irreducible components* of $V(I)$.

3 Let $f(X, Y) = c_{200}X^2 + c_{110}XY + c_{020}Y^2 + c_{101}X + c_{011}Y + c_{002} \in \mathbb{C}[X, Y]$ such that $(c_{200}, c_{110}, c_{020}) \neq (0, 0, 0)$.

a. Show that $V(f) \neq \emptyset$.

b. Let

$$M := \begin{pmatrix} 2c_{200} & c_{110} & c_{101} \\ c_{110} & 2c_{020} & c_{011} \\ c_{101} & c_{011} & 2c_{002} \end{pmatrix}$$

Show that if M is invertible, then f is irreducible.

c. If $\operatorname{rank}(M) \geq 2$ then f is not the square of a liner polynomial (thus $V(f)$ is not a line).

4 Let A be a commutative ring and let $x \in A$. Assume that x is not a zero-divisor but not a unit. Show that for any integer $n \geq 1$, A/xA and A/x^nA have the same associated prime ideals.