D-MATH
HS 2019
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Exercise Sheet 2

Commutative Algebra

Let A be a commutative ring, $I \subseteq A$ an ideal of A and let $S \subseteq A$ be a multiplicative system.

1) Show that A is a local ring with maximal ideal I if and only if $A - A^\times \subseteq I$.

2) Consider the projection map $\pi : A \longrightarrow A/I$. Prove that $\tilde{S} := \pi(S)$ is a multiplicative system of A/I and that there is a ring isomorphism

$$\begin{aligned} S^{-1}A/S^{-1}I &\longrightarrow \tilde{S}(A/I) \\ [a/s] &\longmapsto [a]/[s]. \end{aligned}$$

3) For any prime ideal $\mathfrak{q}$ containing I, there exists a prime ideal $\mathfrak{p}$ of A such that $I \subseteq \mathfrak{p} \subseteq \mathfrak{q}$ and which is minimal among all prime ideals of A containing I.

4) Let $\mathfrak{N}$ be the nilpotent radical of A. Show that the following are equivalent.

   a. A has exactly one prime ideal;
   b. every element of A is either a unit or nilpotent;
   c. $A/\mathfrak{N}$ is a field.

5) Suppose that for each prime ideal $\mathfrak{p}$ of A, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$.
If $A_\mathfrak{p}$ is an integral domain for every $\mathfrak{p}$, is A necessarily an integral domain?

6) Let A be a non-Noetherian ring and let Σ be the set of ideals of A which are not finitely generated. Show that Σ has maximal elements and that the maximal elements of Σ are prime ideals.
Hint: Let $\mathfrak{a}$ be a maximal elements of Σ and suppose that there exist $x, y \in A$ such that $x \notin \mathfrak{a}$, $y \notin \mathfrak{a}$ and $xy \in A$. Show that there exists a finitely generated ideal $\mathfrak{a}_0 \subseteq \mathfrak{a}$ such that $\mathfrak{a}_0 + (x) = \mathfrak{a} + (x)$, and that $\mathfrak{a} = \mathfrak{a}_0 + x(\mathfrak{a} : x)$. Since $(\mathfrak{a} : x) \supset \mathfrak{a}$, it is finitely generated and therefore so is $\mathfrak{a}$.

   Hence a ring in which every prime ideal is finitely generated is Noetherian.