D-MATH
HS 2019
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Exercise Sheet 3

Commutative Algebra

(1) Let (e_1, e_2) be the canonical basis of $\mathbb{R}^2$, and consider the isomorphism of $\mathbb{R}$-vector spaces

$$\begin{aligned} \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 &\xrightarrow{\cong} \mathbb{R}^4 \\ e_1 \otimes e_1 &\longmapsto f_1 \\ e_1 \otimes e_2 &\longmapsto f_2 \\ e_2 \otimes e_1 &\longmapsto f_3 \\ e_2 \otimes e_2 &\longmapsto f_4. \end{aligned}$$

a. Show that $af_1 + bf_2 + cf_3 + df_4 \in \mathbb{R}^4$ is the image of a pure tensor $x \otimes y \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ if and only if

$$ad = bc.$$

b. Compute the matrix $u_1 \otimes u_2$, where $u_1 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -1 & 4 \\ 2 & 3 \end{pmatrix}$.

(2) Let K be a field and let E, F be finitely generated K-vector spaces. We denote by $E' = \mathrm{Hom}_K(E, K)$ and $F' = \mathrm{Hom}_K(F, K)$ the dual space of E and F, respectively. For $x \in E$, $\lambda \in F'$, let $u_{x,\lambda}$ be the linear map

$$\begin{aligned} u_{x,\lambda} : F &\longrightarrow E \\ f &\longmapsto \lambda(f)x. \end{aligned}$$

a. Show that $\{u_{x,\lambda}\}_{x,\lambda}$ is the set of linear maps from F to E of rank ≤ 1.

b. Show that the map

$$\begin{aligned} F' \times E &\longrightarrow \mathrm{Hom}_K(F, E) \\ (\lambda, x) &\longmapsto u_{x,\lambda} \end{aligned}$$

induces an isomorphism

$$F' \otimes_K E \cong \mathrm{Hom}_K(F, E).$$

c. If $\dim F$ and $\dim E$ are ≥ 2, show that not all the elements of $F' \otimes_K E$ are pure tensors.

d. Let $F = E$. Show that there is a unique linear map

$$E' \otimes_K E \longrightarrow K$$

sending $\lambda \otimes x$ in $\lambda(x)$. Using the ismorphism of b., what is the corresponding linear map

$$\mathrm{End}_K(E) \longrightarrow K \text{ ?}$$

e. Find an isomorphism

$$(E \otimes_K F)' \cong E' \otimes_K F'.$$

(3) Let M, N be two A-modules. Without using the explicit construction, show that $M \otimes_A N$ is generated by the elements $\beta(m,n) = m \otimes n$ for $m \in M$, $n \in N$, where

$$\beta : M \times N \longrightarrow M \otimes_A N.$$

Hint: consider the A-submodule T of $M \otimes_A N$ generated by $(\beta(m,n))_{m,n}$.

(4) Let X_1, X_2 be finite sets, and

$$L(X_i) := \{f : X_i \longrightarrow \mathbb{C} \text{ functions}\} \quad i = 1, 2$$

as $\mathbb{C}$-vector spaces (with pointwise operations). Show that the map

$$b : L(X_1) \times L(X_2) \longrightarrow L(X_1 \times X_2)$$

given by

$$b(f_1, f_2)(x, y) = f_1(x) f_2(x)$$

induces an isomorphism of $\mathbb{C}$-vector spaces

$$L(X_1) \otimes_{\mathbb{C}} L(X_2) \longrightarrow L(X_1 \times X_2).$$

(5) Let A be an integral domain and let M be a torsion A-module, i.e. for all $x \in M$, $0 :_A x \neq (0)$. Show that for any A-module N, $M \otimes_A N$ is a torsion A-module.

(6) Let M, N, P be A-modules.

a. Show that there is a unique isomorphism

$$M \otimes_A N \xrightarrow{\iota} N \otimes_A M$$

such that

$$\iota(m \otimes n) = n \otimes m.$$

b. Show that there is a unique isomorphism

$$M \otimes_A (N \otimes_A P) \xrightarrow{\eta} (M \otimes_A N) \otimes_A P$$

such that

$$\eta(m \otimes (n \otimes p)) = (m \otimes n) \otimes p).$$

(7) Show that $2 \otimes 1 = 0$ in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, but $2 \otimes 1 \neq 0$ in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.