D-MATH
HS 2019
Prof. E. Kowalski

Exercise Sheet 5

Commutative Algebra

Let $A \neq 0$ be a commutative ring.

1 Let B be an A-algebra, M a B-module (hence an A-module) and N an A-module.

a. Show that there exists a unique B-module structure on

$$M \otimes_A N$$

such that

$$b(m \otimes n) = bm \otimes n.$$

b. Show that there is a unique isomorphism of B-modules

$$\varphi : M \otimes_A N \longrightarrow M \otimes_B (B \otimes_A N)$$
$$m \otimes n \longmapsto m \otimes (1 \otimes n).$$

2 Let M, N be free A-modules of rank m, n respectively. If there is an isomorphism $\varphi : M \to N$ of A-modules, then prove that $m = n$.
Hint: Consider a maximal ideal $\mathfrak{M}$ of A and the base change to $A/\mathfrak{M}$.

3 Let A be an integral domain and $K = \text{frac(A)}$ its fraction field (so K is an A-algebra).

a. Show that for any K-vector spaces E and F, the K-vector spaces structures on

$$E \otimes_A F$$

defined by

$$\lambda(e \otimes f) = \lambda e \otimes f$$

and

$$\lambda(e \otimes f) = e \otimes \lambda f$$

(for $\lambda \in K$, $e \in E$, $f \in F$) are isomorphic.

b. One has an isomorphism of K-vector spaces

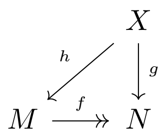
$$\varphi : E \otimes_K F \longrightarrow E \otimes_A F$$
$$e \otimes_K f \longmapsto e \otimes_A f.$$

(4) Let M, N, L be A-modules. Find a natural isomorphism of A-modules

$$\varphi : \mathrm{Hom}_A(M \otimes_A N, L) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, L))$$

("Hom-tensor adjunction formula").

(5) An A-module X is called **projective** if for every M, N A-module, for every surjective A-linear map $M \xrightarrow{f} N$ and every A-linear map $X \xrightarrow{g} N$, there exists an A-linear map $X \xrightarrow{h} M$ such that the following diagram commutes:



Show the following facts.

a. A free A-module is a projective A-module.

b. For any A-module M, there exists a projective module X and a surjective map
$$\varepsilon : X \longrightarrow M.$$

c. $\mathbb{Q}$ as $\mathbb{Z}$-module is not projective.
Hint: Consider $N = \mathbb{Q}$, $g = \mathrm{id}_{\mathbb{Q}}$, $M = \oplus_{n>0}\mathbb{Z}$ the $\mathbb{Z}$-free module with basis $(e_n)_{n>0}$, $f =?$