D-MATH
HS 2019
Prof. E. Kowalski

Exercise Sheet 6

Commutative Algebra

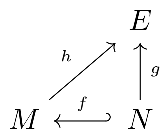
Let $A \neq 0$ be a commutative ring.

1 A short exact sequence is **split** if it is isomorphic to

$$0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0$$

for some A-modules M, N. Find an example of non-split exact sequence.

2 An A-module E is called **injective** if for every M, N A-modules, for every injective A-linear map $N \xrightarrow{f} M$ and every A-linear map $N \xrightarrow{g} E$, there exists an A-linear map $M \xrightarrow{h} E$ such that the following diagram commutes:



An A-module X is called **divisible** in for all $x \in X$ and for all non-zero divisor $a \in A$, there exists $x' \in X$ such that

$$x = ax'.$$

a. Prove that if E is injective, then E is divisible.

b. If moreover A is a PID, then

$$E \text{ injective} \Longleftrightarrow E \text{ divisible}.$$

c. Find examples of injective and a non-injective modules.

3 Let A be a PID and let M be a finitely generated A-module.

a. Show that there exist integers $m, n \geq 0$ and a free resolution

$$0 \longrightarrow A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0.$$

b. There exists a resolution of the form

$$0 \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

if and only if M is A-free.

c. Compute a sequence in a. for $A = \mathbb{Z}$, $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}$.

d. Compute a sequence in a. for $A = \mathbb{Q}[X]$, $M = \mathbb{Q}[X]/(X^2 + 1)$.

(4) Prove the "Five lemma": let M_i, N_i, $i = 1, \dots, 5$ be A-modules. Consider the following diagram with exact rows

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \xrightarrow{f_3} & M_4 & \xrightarrow{f_4} & M_5 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\ N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & \xrightarrow{g_3} & N_4 & \xrightarrow{g_4} & N_5 \end{array}$$

Then:

a. h_2, h_4 surjective, h_5 injective $\Longrightarrow h_3$ surjective;

b. h_2, h_4 injective, h_1 surjective $\Longrightarrow h_3$ injective;

c. $h_1, \dots, h_5$ isomorphisms $\Longrightarrow h_3$ isomorphism.

(5) Let K be a field, $n \geq 0$, $E_1, \dots, E_n$ finitely dimensional K-vector spaces. If

$$0 \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow 0$$

is an exact sequence of K-vector spaces, then

$$\sum_{i=1}^{n} (-1)^i \dim E_i = 0.$$

(6) Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of A-modules. Show that if M', M'' are finitely generated, then so is M.

(7) Let $m \leq n$ be positive integers and let $A = \mathbb{Q}[X]/(X^n)$, $M = A/(X^m)$. Show that M has an infinite free resolution, as A-module.